Equivalence of the different definitions of the Sasaki distance

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Abstract

Different notions of a distance can be defined on the unit tangent bundle of the hyperbolic plane. Here we prove by elementary methods that a geometric definition and a dynamical definition distance are equivalent.

We denote the hyperbolic distance on the hyperbolic plane $\mathbb{H}$ by d. For $v \in T^1\mathbb{H}$, c_v is the unit speed geodesic with $c_v'(0) = v$. Let $v, w \in T^1\mathbb{H}$. We define

$$d_1(v, w) := d(c_v(0), c_w(0)) + d(c_v(1), c_w(1)).$$

Let $c_{v,w}$ be the geodesic passing through the base points of v and w. Let $\alpha, \beta \in [-\pi, \pi)$ be the oriented angles formed by v and w with the geodesic $c_{v,w}$, respectively. We define

$$d_2(v, w) := d(c_v(0), c_w(0)) + |\alpha - \beta|.$$

Proposition 0.1. *Let $v, w \in T^1\mathbb{H}$. We set $D_0 := d(c_v(0), c_w(0))$, $D_1 := d(c_v(1), c_w(1))$, $S := \sinh(1)$ and $C := \cosh(1)$. They satisfy*

$$\begin{aligned}\cosh D_1 =& C^2 \cosh D_0 - 2CS \sinh D_0 \sin(\frac{\alpha-\beta}{2}) \sin(\frac{\alpha+\beta}{2}) \\ &- S^2 \cosh^2(\frac{D_0}{2}) \cos(\alpha - \beta) - S^2 \sinh^2(\frac{D_0}{2}) \cos(\alpha + \beta).\end{aligned} \quad (1)$$

Proof. We consider the triangle T_1 with vertices $c_v(0), c_w(0)$ and $c_v(1)$ and the triangle T_2 with vertices $c_w(0), c_v(0$ and $c_w(1)$. Let β' the interior oriented angle of T_1 at $c_w(0)$. T_1 has sides of length D_0, 1 and $D_2 := d(c_v(1), c_w(0))$. Applying the hyperbolic cosine rule for T_1, we have

$$\cosh D_2 = C \cosh D_0 - S \sinh D_0 \cos(\pi - \alpha), \quad (2)$$

$$C = \cosh D_0 \cosh D_2 - \sinh D_0 \sinh D_2 \cos \beta'. \quad (3)$$

By the sine rule between the angles at points $c_v(0)$ and $c_w(0)$, we also have

$$\sinh D_2 \sin \beta' = S \sin(\pi - \alpha). \quad (4)$$

The angle of T_2 at $c_w(0)$ is $\beta - \beta'$. By the hyperbolic cosine rule for T_2,

$$\cosh D_1 = C \cosh D_2 - S \sinh D_2 \cos(\beta - \beta'). \quad (5)$$

Replacing (2) in (3), we obtain

$$C = C \cosh^2 D_0 + S \cosh D_0 \sinh D_0 \cos \alpha - \sinh D_0 \sinh D_2 \cos \beta',$$

which can be rewritten as

$$\sinh D_2 \cos \beta' = C \sinh D_0 + S \cosh D_0 \cos \alpha. \quad (6)$$

By a trigonometric relation at (5) and then replacing (2), (4) and (6), we have

$$\begin{aligned}\cosh D_1 &= C\cosh D_2 - S\sinh D_2\cos\beta\cos\beta' - S\sinh D_2\sin\beta\sin\beta' \\ &= C^2\cosh D_0 + CS\sinh D_0\cos\alpha - CS\sinh D_0\cos\beta \\ &- S^2\cosh D_0\cos\alpha\cos\beta - S^2\sin\alpha\sin\beta.\end{aligned}$$

Elementary trigonometric manipulations lead to the formula of the statement. □

Lemma 0.2. *There exists $\varepsilon > 0, K > 0$ such that, for all $\alpha, \beta \in \mathbb{R}$, $D_0, D_1 \in \mathbb{R}_+$ satisfying Equation (1), if $D_0 + |\alpha - \beta| < \varepsilon$ then*

$$D_1 \leq K(D_0 + |\alpha - \beta|)$$

and if $D_0 + D_1 < \varepsilon$ then

$$|\alpha - \beta| \leq D_0 + D_1.$$

Proof. We can compare the Taylor polynomials of order two at $D_1 = D_0 = \alpha - \beta = 0$ of both sides of (1). We have

$$\begin{aligned}1 + \frac{D_1^2}{2} + o_2(D_1) &= C^2(1 + \frac{D_0^2}{2}) - CSD_0(\alpha-\beta)\sin(\frac{\alpha+\beta}{2}) \\ &- S^2(1 + \frac{D_0^2}{8})^2(1 - \frac{(\alpha-\beta)^2}{2}) - \frac{1}{4}S^2 D_0^2\cos(\alpha+\beta) + o_2(D_0, \alpha-\beta) \\ &= 1 + \frac{1}{2}\left(C^2 - S^2\frac{1+\cos(\alpha+\beta)}{2}\right)D_0^2 - CS\sin\left(\frac{\alpha+\beta}{2}\right)D_0(\alpha-\beta) \\ &+ \frac{1}{2}S^2(\alpha-\beta)^2 + o_2(D_0, \alpha-\beta).\end{aligned}$$

The polynomial of the right hand side is bounded below by $1+\frac{S^2}{2}(|\alpha-\beta|-D_0)^2$ and above by $1+\frac{C^2}{2}(|\alpha-\beta|+D_0)^2$. Now we choose constants K, K' such that $1 < K' < S < C < K$. There exists $\varepsilon > 0$ depending only on K and K' small enough so higher order terms can be neglected: if $D_0 + |\alpha - \beta| < \varepsilon$ then

$$1 + \frac{D_1^2}{2} \leq 1 + \frac{K^2}{2}(|\alpha-\beta| + D_0)^2$$

and if $D_0 + D_1 < \varepsilon$

$$1 + \frac{K'^2}{2}(|\alpha-\beta| - D_0)^2 \leq 1 + \frac{D_1^2}{2}.$$

The first inequality is equivalent to $D_1 \leq K(D_0+|\alpha-\beta|)$ and the second one is equivalent to $|\alpha - \beta| \leq D_0 + \frac{1}{K'}D_1$, which implies the statement. □

Proposition 0.3. *The distances d_1 and d_2 are equivalent.*

Proof. We have $D_1 \leq d(c_v(0), c_v(1)) + D_0 + d(c_w(0), c_w(1))$. If $D_0 + |\alpha - \beta| \geq \varepsilon$, then

$$D_1 \leq D_0 + \frac{2}{\varepsilon}(D_0 + |\alpha - \beta|)$$

In view of Lemma 0.2, we have

$$d_1(v,w) \leq D_0 + \max(D_0 + \frac{2}{\varepsilon}(D_0 + |\alpha-\beta|), K(D_0+|\alpha-\beta|)) \leq \max(2+\frac{2}{\varepsilon}, K+1)d_2(v,w).$$

On the other hand, if $D_0 + D_1 \geq \varepsilon$, then $|\alpha - \beta| \leq 2\pi \leq \frac{2\pi}{\varepsilon}(D_0 + D_1)$. Therefore

$$d_2(v,w) \leq D_0 + \max(\frac{2\pi}{\varepsilon}, 1)(D_0 + D_1) \leq \max(1 + \frac{2\pi}{\varepsilon}, 2)d_1(v,w).$$

□