Geodesic flow in negative curvature

Master's thesis

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Chapter 1

Introduction

The object of this master's thesis is the study of dynamical properties of the geodesic flow in negative curvature. This flow acts on the tangent bundle of a Riemannian manifold by moving tangent vectors to a geodesic a certain amount of time along the geodesic itself. The geometry of negative curvature provides some interesting dynamical properties to the flow, for example, the ergodic property. It is a classical problem in the theory of dynamical systems.

One of the first results is due to E. Hopf, who proved the ergodicity of the geodesic flow on manifolds of curvature -1 with finite volume [Hop36]. Later it was also proved that the geodesic flow in this context is mixing. D. V. Anosov proved a generalization of the ergodicity of the geodesic flow to negative curvature [Ano67]. We will study both situations: the case of curvature -1 for surfaces and the case of variable curvature and any dimension.

The text is divided in four chapters, the first of them being this introduction. The second is also an introductory chapter to the objects that we will study, manifolds of negative curvature. We put special emphasis to the hyperbolic plane and we describe the surfaces of curvature -1. In Chapter 3, we introduce the geodesic flow and another kind of flow, called horocyclic, on the hyperbolic plane and we study their dynamics. The proof of their dynamical properties is based on the conjugation with an algebraic model, that is easier to study. In the last chapter, we show the ergodicity in variable curvature. We define some tools and state some results of geometry in negative curvature that is needed and then we proceed to give evidence of the ergodicity, going through some rather technical aspects of geodesic flow.

All the results here presented were already known, although it does contain some own computations. The main job done on this master's thesis consisted in learning all the new concepts and gathering results from different sources to make a coherent explanation of the subject. The ergodicity in negative curvature ultimately depends on some facts concerning spaces of negative curvature. Some of them are out of topic, but would be an excellent way to make our understanding better beyond this master's thesis.

Geodesic and horocyclic flow are known to satisfy stronger dynamical properties than the ones we will see. The domain that studies these flows has still a lot of open questions and some properties admit generalizations in some sense. For example, in the case of nonpositive curvature, it is not clear yet what happens with a property called equidistribution.

Chapter 2

Manifolds of negative curvature

The goal of this chapter is to give the necessary background concerning the objects we will study. We present well-known results of Riemannian geometry in negative curvature (Section 2.1), putting particular attention to the case of constant curvature, where we are able to describe the complete Riemannian manifolds (Section 2.2). Section 2.3 is dedicated to the hyperbolic plane and its isometries. We introduce the discrete groups of isometries, which play an important role in the next chapter.

We do not present the proof of general results in Riemannian geometry, they can be found in [dC92]. In the study of isometries of hyperbolic plane, [Dal11] and [Kat92] are good references, from which we have taken inspiration.

2.1 The theorem of Hadamard

The sectional curvature $K(x, \Pi)$ of a Riemannian manifold M^n, $n \geq 2$ is defined at each point $x \in M$, for each 2-dimensional subspace $\Pi \subset T_xM$. In this text, we are interested in manifolds with negative curvature, that is to say, negative at every point and every plane. Here, we present a result on manifolds of nonpositive curvature.

We say that M is *(geodesically) complete* if the geodesics are defined for all time. We assume the reader has notions on complete manifolds and the Theorem of Hopf-Rinow, which gives equivalent conditions to the fact of being complete and important properties of the geodesics. As a consequence, we deduce that every compact Riemannian manifold is complete.

This short section finishes with the statement of the theorem, which is proved using Hopf-Rinow.

Theorem 1. *(Hadamard). Let M^n be a complete Riemannian manifold of nonpositive sectional curvature. Then, for all x in M, the map $\exp_x : T_xM \to M$ is a covering map. In particular, the universal cover of M is diffeomorphic to $\mathbb{R}^n$.*

2.2 Manifolds of constant curvature

Now we restrict our attention to spaces of constant curvature. First of all, we remark that, in a manifold M with a Riemannian metric g, a rescaling of the metric $h = \lambda g$, $\lambda > 0$ changes the sectional curvature as $K_h = \frac{1}{\lambda} K_g$. Thanks to these rescalings we can focus to what happens in curvatures -1, 0 and 1. Although we are only interested in the first case, we will explain the others for sake of completeness.

We introduce a well known space for each of these curvatures. Consider the submanifold $\mathbb{H}^n$ of $\mathbb{R}^n$ equipped with the metric g_{-1}, where

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n > 0\}, \quad g_{-1} = \frac{1}{x_n^2}\left(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2\right).$$

With some computations, it is easy to see that its sectional curvature is constant -1. With curvature 0, we take $\mathbb{R}^n$ with the Euclidean metric g_0 and with curvature 1, the sphere $\mathbb{S}^n$ with the inherited metric g_1 from $\mathbb{R}^{n+1}$. The geodesic trajectories of the two last spaces are easy to find, they are straight lines in the euclidean space and great circles in the sphere.

Proposition 1. *The geodesic trajectories of* $(\mathbb{H}^n, g_{-1})$ *are the straight lines perpendicular to the hyperplane* $\{x_n = 0\}$ *and the circles with center in* $\{x_n = 0\}$ *and perpendicular to it.*

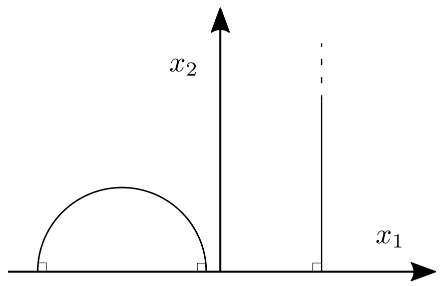

Figure 2.1: Two geodesics on the hyperbolic plane.

In Figure 2.1 we see a vertical and a half-circle geodesic on the hyperbolic plane. We conclude that all three spaces are simply connected, geodesically complete and have constant curvature. The following result is what makes them particularly interesting.

Theorem 2. *Let* M^n *be a complete Riemannian manifold of constant sectional curvature* $K = -1, 0, 1$. *Then, the universal cover* $\tilde{M}$ *of* M*, with the lifted metric, is isometric to:*

(i) $(\mathbb{H}^n, g_{-1})$ *if* $K = -1$,

(ii) $(\mathbb{R}^n, g_0)$ *if* $K = 0$,

(iii) $(\mathbb{S}^n, g_1)$ *if* $K = 1$.

We can go further in our attempt to understand all the spaces of constant curvature.

Definition 1. Let M be a Riemannian manifold and Γ a subgroup of isometries of M. We say that Γ acts *totally discontinuously* if for all x in M there is a neighborhood U of x such that $\gamma(U) \cap U = \emptyset$ for all γ in $\Gamma \setminus \{id\}$.

We say that Γ acts *properly discontinuously* if for all x in M there is a neighborhood U of x such that $\gamma(U) \cap U = \emptyset$ for all but finitely many γ in Γ.

It is clear that the first property implies the second one, but it is convenient to introduce both of them. They have the goal to provide nice properties on the quotient of the set M under the action by evaluation of Γ.

In an introductory course in Riemannan geometry, it is seen that, given a Riemannian manifold M and a group Γ of isometries acting totally discontinuously, we can put a natural structure of Riemannian manifold on the set

$$M/\Gamma = \{\Gamma x := \{\gamma(x) \mid \gamma \in \Gamma\} \mid x \in M\}\,.$$

Since M and M/Γ are locally isometric, if M has constant curvature, M/Γ will also have constant curvature with the same value. By taking $M = \mathbb{R}^n$, $\mathbb{H}^n$, $\mathbb{S}^n$ and a suitable group of isometries Γ, we can generate lots of manifolds of constant curvature. In fact, all complete manifolds of constant curvature have this form.

Proposition 2. *Let M^n be a complete Riemannian manifold of constant sectional curvature $K = -1, 0, 1$. Then, M is isometric to the quotient $\tilde{M}/\Gamma$ of the universal cover $\tilde{M}$ of M, with the lifted metric, by a subgroup of isometries of $\tilde{M}$ that acts totally discontinuously.*

The proposition reduces the problem of finding manifolds with curvature -1 to the study of subgroups of isometries of $\mathbb{H}^n$ that act totally discontinuously. In Section 2.3, we will describe these groups in dimension 2 and, in consequence, the surfaces with curvature -1.

In fact, we will study properly discontinuous subgroups, because all the work that we will do in Chapter 3 is valid for this kind of subgroups. In general, the quotient M/Γ by a properly discontinuous group is no longer a manifold, but it is an example of a more general object called an orbifold.

2.3 Surfaces of curvature -1

In this section, we take a close look at the geometry of surfaces with curvature -1. To simplify notations, the hyperbolic half-plane will be simply denoted by $\mathbb{H}$, with coordinates (x, y), and the metric given by $g = \frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. We also identify $\mathbb{H} \equiv \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ via $z = x + iy \equiv (x, y)$.

Homographies are maps φ from $\mathbb{C} \cup \{\infty\}$ to itself of the form

$$\varphi(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C},\ ad - bc \neq 0. \tag{2.1}$$

They have the property to send the set of lines and circles to itself. Using homographies, we can smoothly transform the upper half-plane $\mathbb{H}$ to other domains in $\mathbb{C}$ and pushing forward the metric of $\mathbb{H}$ we obtain an isometric space. This means that there are many other models for hyperbolic geometry. One of them is the Poincaré disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ with the metric given by

$$\frac{4}{(1 - |z|^2)^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right).$$

The transformation between them is

$$\begin{array}{rccc} \Psi: & \mathbb{H} & \longrightarrow & \mathbb{D} \\ & z & \longmapsto & \frac{iz+1}{z+i} \end{array}.$$

This model is useful because of its symmetry, certain geometric reasonings are easier to visualize. Recall that geodesics in the half-plane are vertical lines and semicircles with center on the real axis (thus, perpendicular to it). Since the previous transformation is a homography that sends i to 0, geodesics in the disk are circles perpendicular to the border of the disk and lines passing through 0.

Now, we study the group of isometries $\operatorname{Isom}(\mathbb{H})$ of the hyperbolic half-plane. Consider a homography φ of the form (2.1) with real coefficients a, b, c, d. We obtain by computation

$$\operatorname{Im}\varphi(z) = \frac{\operatorname{Im} z}{|cz+d|^2}, \quad \mathrm{d}\varphi = \frac{1}{(cz+d)^2}\mathrm{d}z.$$

The first implies that φ is well defined as a bijective map $\mathbb{H} \to \mathbb{H}$. The tangent map of such an homography is $d_z\varphi(v) = \varphi'(z) \cdot v$, where z is in $\mathbb{H}$ and $v = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} \in T_z\,\mathbb{H}$ is identified with the complex number $v_x + iv_y$. The determinant of the tangent map is $|\varphi'(z)|^2$. It is satisfied

$$g(d_z\varphi(v), d_z\varphi(v)) = \frac{|\mathrm{d}\varphi(v)|^2}{\operatorname{Im}(\varphi(z))^2} = \frac{|\mathrm{d}z|^2}{\operatorname{Im}(z)^2} = g(v,v),$$

so φ turns to an orientation-preserving isometry.

This type of homographies are identified with $\mathrm{PSL}_2(\mathbb{R})$ by a group isomorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(z \mapsto \frac{az+b}{cz+d} \right).$$

We will refer to them as homographies associated to $\mathrm{PSL}_2(\mathbb{R})$ and when it is clear from the context we will directly write $\mathrm{PSL}_2(\mathbb{R})$ for the subgroup of homographies.

Proposition 3. *All the orientation-preserving isometries of* $\mathbb{H}$ *are homographies associated to* $\mathrm{PSL}_2(\mathbb{R})$.

Proof. Consider $\mathbb{H} \subset \mathbb{R}^2$ equipped with the standard metric $\langle\,\cdot\,,\cdot\,\rangle$. A diffeomorphism $\phi : \mathbb{H} \to \mathbb{H}$ is conformal if it preserves the orientation and there exits a function $f : \mathbb{H} \to (0, +\infty)$ such that $\langle d_z\phi(v), d_z\phi(v)\rangle = f(z)\langle v, v\rangle$. We observe that orientation-preserving isometries with the metric g are conformal. Let us prove that conformal maps of $\mathbb{H}$ are homographies associated to $\mathrm{PSL}_2(\mathbb{R})$.

We will use the fact that conformal maps between two open sets are holomorphic diffeomorphisms. In addition, holomorphic diffeomorphisms are automatically biholomorphisms.

Since the transformation Ψ is holomorphic, conformal maps of the disk are $\operatorname{Conf}(\mathbb{D}) = \Psi\operatorname{Conf}(\mathbb{H})\Psi^{-1}$. Therefore, proving that $\operatorname{Conf}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R})$ is equivalent to proving that $\operatorname{Conf}(\mathbb{D}) = \Psi\,\mathrm{PSL}_2(\mathbb{R})\Psi^{-1}$. It can be seen that the set equality

$$\Psi\,\mathrm{PSL}_2(\mathbb{R})\Psi^{-1} = \left\{ h_{\alpha,\beta} \,\middle|\, \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \right\}$$

is held, where

$$h_{\alpha,\beta}(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}.$$

So, to finish the proof it is enough to show that every biholomorphism of the disk 0 is of the form $h_{\alpha,\beta}$.

This is an application of Schwarz Lemma. Suppose the map $\phi : \mathbb{D} \to \mathbb{D}$ is biholomorphic. There exist $h_{\alpha,\beta}$ which sends $\phi(0)$ to 0. By the Schwarz Lemma, since $h_{\alpha,\beta}\phi$ fixes 0, we obtain

$$|h_{\alpha,\beta}\phi(z)| \leq |z| \text{ and } \left|(h_{\alpha,\beta}\phi)^{-1}(z)\right| \leq |z|, \quad \forall z \in \mathbb{D},$$

so $|h_{\alpha,\beta}\phi(z)| = |z|$. Again by Schwarz Lemma, $h_{\alpha,\beta}\phi(z) = az$ for some complex number $a \in \mathbb{C}$, $|a| = 1$, which implies that the map ϕ is as wanted. $\square$

It is clear that orientation-preserving isometries of $\mathrm{PSL}_2(\mathbb{R})$ is a subgroup of index 2 of $\mathrm{Isom}(\mathbb{H})$, this makes them the main ingredient in the study of isometries of $\mathbb{H}$ and, thus, of surfaces with curvature -1.

A discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ (with the topology of a quotient of a subspace of the space of matrices) is called a *Fuchsian group.* We will show that the subgroups of $\mathrm{PSL}_2(\mathbb{R})$ acting properly discontinuously on $\mathbb{H}$ are precisely the discrete subgroups. We start with some alternative characterizations of properly discontinuous actions.

Proposition 4. *Let M be a Riemannian manifold and Γ a subgroup of isometries of M. The following are equivalent:*

(i) The subgroup Γ acts properly discontinuously.

(ii) For all x in M, for all compact subset K of M, the set

$$\{\gamma \in \Gamma \,|\, \gamma(x) \in K\}$$

is finite.

(iii) For all x in M, the set Γx is discrete and the stabilizer at the point x is finite.

Proof. (ii) $\Longrightarrow$ (iii). Take a compact neighborhood K of x. The stabilizer at x is the set $\{\gamma \in \Gamma \,|\, \gamma(x) = x\}$, which is included in the set $\{\gamma \in \Gamma \,|\, \gamma(x) \in K\}$, and it is clear that it has to be finite. Moreover, the set $\Gamma x \cap K$ is included in the set $\{\gamma \in \Gamma \,|\, \gamma(x) \in K\}\,(x)$, so it is finite. Then, there is a neighborhood of x which does not contain the points of Gx different from x.

(iii) $\Longrightarrow$ (ii). Let K be a compact set. The intersection $\Gamma x \cap K$ is finite. The cardinal of the set $\{\gamma \in \Gamma \,|\, \gamma(x) \in K\}$ is the sum of the cardinals of the stabilizers at y over all y $\Gamma x \cap K$, so it is finite.

(iii) $\Longrightarrow$ (i). Let K be a compact set. The intersection $\Gamma x \cap K$ is finite. Then, there exists a ball $B(x, \varepsilon)$ centered at x of radius ε which does not contain any of the points in Γx different from x. It follows that $\gamma(B(x, \varepsilon/2)) \cap B(x, \varepsilon/2) \neq \emptyset$ implies that the isometry γ is in the stabilizer at the point x, but this can only happen for a finite amount of γ in Γ.

(i) $\Longrightarrow$ (iii). Let x in M. For each neighborhood V of x, we have

$$\{\gamma \in \Gamma \,|\, \gamma(x) = x\} \subset \{\gamma \in \Gamma \,|\, \gamma(V) \cap V \neq \emptyset\}\,.$$

and by hypothesis of properly discontinuity we can choose one neighborhood V such that the second set is finite, so the stabilizer at x is finite as well.

For the same choice of V, we look at the set $\Gamma x \cap V$. If some point y is in $\Gamma x \cap V$, we can write $y = \gamma(x)$ for some homography γ in Γ, and it satisfies $\gamma(V) \cap V \neq \emptyset$. Since there are only finitely many γ that satisfy that, the set $\Gamma x \cap V$ is finite, so we can find a smaller neighborhood of x which contains no point of Γx other than x.

□

To prove the equivalence between discrete and properly discontinuous we need the next lemma.

Lemma 1. *Let $z \in \mathbb{H}$ and let K be a compact subset of $\mathbb{H}$. Then, the set*

$$\{\gamma \in \mathrm{PSL}_2(\mathbb{R}) \,|\, \gamma(z) \in K\}$$

is compact.

The proof of this lemma can be found in [Kat92]. With its help we prove the following result.

Theorem 3. *A subgroup* Γ *of* $\mathrm{PSL}_2(\mathbb{R})$ *is discrete if and only if it acts properly discontinuously on* $\mathbb{H}$.

Proof. Suppose the subgroup Γ is discrete. We will use that Γ is closed, it is a fact from general topology that a discrete subgroup of a Hausdorff topological group is closed. For all x in $\mathbb{H}$, for all compact set K, the set

$$\{\gamma \in \Gamma \,|\, \gamma(x) \in K\} = \{\gamma \in \mathrm{PSL}_2(\mathbb{R}) \,|\, \gamma(x) \in K\} \cap \Gamma$$

is discrete and compact, because it is the intersection of a compact set (by Lemma 1) with a closed set. Then, it has to be finite, so the action of Γ is properly discontinuous.

Conversely, suppose that Γ is properly discontinuous. Let us suppose that Γ is not discrete and obtain a contradiction. There exist γ in Γ and distinct elements γ_n in Γ, for all n in $\mathbb{N}$, such that $\lim \gamma_n = \gamma$. Let z in $\mathbb{H}$. Then, the sequence $\gamma^{-1}\gamma_n(z) \to z$ converges. In consequence, for any neighborhood V of z, there exists n_0 in $\mathbb{N}$ such that $\gamma^{-1}\gamma_n(V) \cap V \neq \emptyset$ if $n \geq n_0$. This contradicts that Γ is properly discontinuous. $\square$

Chapter 3

Geodesic and horocyclic flows in curvature -1

In this chapter, we will introduce the geodesic flow and the horocyclic flows for surfaces of curvature -1, which have been an important object of study in classical dynamical systems. In Section 3.1 we will present the two flows in a very algebraic context, where they act by matrix multiplications. Thanks to some explicit relations between them it will be easy to prove the dynamical properties of these flows, ergodicity and mixing, on convenient spaces. In Section 3.2 we define the flows geometrically on the hyperbolic plane and we establish the equivalence with the first model. In Chapter 4 we will deal with a more general situation and we will use a different method to study the dynamics. However, this other method was first applied in curvature -1, and we expect that the connection between both cases will be clear thanks to the notions here explained. The chapter is based on the notes from Y. Coudène [Cou14].

3.1 The algebraic model

3.1.1 Dynamics on $\mathrm{PSL}_2(\mathbb{R})$

The space where we will work is $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\{\pm Id\}$, where

$$\mathrm{SL}_2(\mathbb{R}) = \{A \in \mathrm{M}_2(\mathbb{R}) \mid \det A = 1\}.$$

This set has a structure of differential manifold as a submanifold of the set of matrices and it can be easily described with two coordinate charts, defined on the open sets

$$U_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}) \,\middle|\, d > 0 \right\},$$

$$U_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R}) \,\middle|\, c > 0 \right\},$$

which cover the whole manifold $\mathrm{PSL}_2(\mathbb{R})$. Explicitly, the charts are

$$\begin{array}{cccc} \psi_1 : & U_1 & \longrightarrow & \mathbb{R} \times \mathbb{R} \times (0, +\infty) \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & (b, c, d) \end{array},$$

$$\begin{array}{cccc} \psi_2 : & U_2 & \longrightarrow & \mathbb{R} \times (0, +\infty) \times \mathbb{R} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & (a, c, d) \end{array}.$$

Let us define a very special measure μ on $\mathrm{PSL}_2(\mathbb{R})$, called the Liouville measure. The measure is defined from its densities on each chart:

$$\mathrm{d}\mu = \frac{1}{d}\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}d \text{ on } U_1 \text{ and } \mathrm{d}\mu = \frac{1}{c}\,\mathrm{d}a\,\mathrm{d}c\,\mathrm{d}d \text{ on } U_2.$$

A little computation shows that the Jacobian of the change of coordinates from U_1 to U_2 is $|c/d|$, so the two definitions of $\mathrm{d}\mu$ coincide on $U_1 \cap U_2$ and we get a globally defined measure.

Theorem 4. *The measure μ is invariant by left and right multiplication on* $\mathrm{PSL}_2(\mathbb{R})$.

Proof. The measure μ is invariant by a differentiable function f from $\mathrm{PSL}_2(\mathbb{R})$ to itself if $\mu(U) = \mu(f^{-1}(U))$ for all open set U, or equivalently, the equality of differential forms $f^*\mathrm{d}\mu = \mathrm{d}\mu$ is satisfied. Suppose that f is a left multiplication, i. e. there is g in $\mathrm{PSL}_2(\mathbb{R})$ such that $f(A) = gA$ for all A in $\mathrm{PSL}_2(\mathbb{R})$. Writing $g = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$, we have

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{pmatrix},$$

which in coordinates of U_1 is

$$f(b, c, d) = (\alpha b + \beta d, \gamma\frac{1 + bc}{d} + \delta c, \gamma b + \delta d) = (b', c', d').$$

Then, the Jacobian is

$$j = \left|\det\begin{pmatrix} \alpha & * & \beta \\ 0 & \gamma\frac{b}{d} + \delta & 0 \\ \gamma & * & \delta \end{pmatrix}\right| = \left|(\alpha\delta - \beta\gamma)\frac{\gamma b + \delta d}{d}\right| = \left|\frac{d'}{d}\right|$$

and from $\mathrm{d}b'\,\mathrm{d}c'\,\mathrm{d}d' = j\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}d$ we obtain

$$f^*\mathrm{d}\mu = \frac{\mathrm{d}b'\,\mathrm{d}c'\,\mathrm{d}d'}{d'} = \frac{\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}d}{d} = \mathrm{d}\mu.$$

The computations done so far are sufficient since $\mathrm{PSL}_2(\mathbb{R}) \setminus U_1$ has zero measure. The proof for right multiplication goes similarly. □

The *geodesic flow* $\{g_t\}_{t\in\mathbb{R}}$ is a flow on $\mathrm{PSL}_2(\mathbb{R})$ defined by

$$g_t(M) = M\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

There are also the *contracting horocyclic flow* $\{h_s^+\}_{s\in\mathbb{R}}$ and the *expanding horocyclic flow* $\{h_u^-\}_{u\in\mathbb{R}}$ defined by

$$h_s^+(M) = M\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \qquad h_u^-(M) = M\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

From Theorem 4, μ is invariant by the geodesic (and horocyclic) flow in the sense that it is invariant by g_t (or $h_t^\pm$) for each t in $\mathbb{R}$.

These flows satisfy some relations between them that can be showed by straight computation, but they will be crucial to show the dynamical properties of the flows.

Proposition 5. *The geodesic and horocyclic flows satisfy:*

$$g_t \circ h_s^+ = h_{se^{-t}}^+ \circ g_t, \qquad s,t \in \mathbb{R}, \tag{3.1}$$

$$g_t \circ h_u^- = h_{ue^t}^- \circ g_t, \qquad t,u \in \mathbb{R}, \tag{3.2}$$

$$h^+_{\frac{s^{-1}-1}{\varepsilon}} \circ h^-_{\varepsilon} \circ h^+_{\frac{s-1}{\varepsilon}} \circ h^-_{-\varepsilon s^{-1}} = g_{2\log s}, \qquad s,\varepsilon > 0. \tag{3.3}$$

Moreover, they generate the action of $\mathrm{PSL}_2(\mathbb{R})$ *by right multiplication, i. e. if* $A \in \mathrm{PSL}_2(\mathbb{R})$*, the right multiplication by* A*,* R_A*, is*

$$R_A = \begin{cases} h^-_{c/d} \circ g_{-2\log d} \circ h^+_{b/d} & \textit{if } d > 0, \\ h^-_{c+a/b} \circ h^+_b \circ h^-_{-1/b} & \textit{if } d = 0, \end{cases}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

3.1.2 Quotients of finite volume

The space $\mathrm{PSL}_2(\mathbb{R})$ acts as the cover of smaller spaces that interest us. In the following we formalize this notion. Let Γ be a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ which is discrete as a subset, acting on $\mathrm{PSL}_2(\mathbb{R})$ by left multiplication. We will refer to such a Γ as a *discrete subgroup*. Denote by $X = \Gamma\backslash \mathrm{PSL}_2(\mathbb{R})$ the group quotient and π the projection to the quotient.

Definition 2. We say that a discrete subgroup Γ has *finite covolume* or that X has *finite volume* if there exists a Borel subset D of $\mathrm{PSL}_2(\mathbb{R})$ that satisfies:

(i) $\mu(D) < +\infty$, $\mu(\partial D) = 0$,

(ii) $\mathrm{PSL}_2(\mathbb{R}) = \coprod_{\gamma\in\Gamma} \gamma D$.

Suppose the group Γ has finite covolume. The Liouville measure on $\mathrm{PSL}_2(\mathbb{R})$ induces a measure $\overline{\mu}$ on X, defined by

$$\overline{\mu}(A) = \mu(\pi^{-1}(A) \cap D).$$

It is possible that more than one set D satisfies the conditions in the definition. Nevertheless, the measure does not depend on the choice of D.

Right multiplications on $\mathrm{PSL}_2(\mathbb{R})$ commute with left ones, so the flows g_t, h_s^+, h_u^- induce flows on X, because it is a left quotient. We keep the same names and notation for these new flows. To be consistent we will use μ for the measure induced on X. Because of the left and right multiplication invariance of μ on $\mathrm{PSL}_2(\mathbb{R})$, the geodesic and horocyclic flows on X are μ-invariant.

3.1.3 Dynamical properties

It is time to introduce the two dynamical properties that will be studied. We define them for a general (finite) measure preserving flow $(X, \mathcal{A}, \mu, \{\Phi^t\}_{t\in\mathbb{R}})$, where $(X, \mathcal{A}, \mu)$ is a finite measure space and Φ^t a measurable flow on X that is μ-invariant. A function $f \in L^2(x,\mu)$ is Φ^t-invariant if $f \circ \Phi^t = f$ a.e. for all $t \in \mathbb{R}$.

Definition 3. Let $(X, \mathcal{A}, \mu, \{\Phi^t\}_{t\in\mathbb{R}})$ be a measure preserving flow. We say that Φ^t is *ergodic with respect to* μ if every invariant function $f \in L^2(X,\mu)$ is constant a.e.

There are several equivalent definitions of ergodicity, for instance, we can say that a flow is ergodic if all invariant sets are either negligible or of negligible complement. A measurable set A is invariant if $\mu(\Phi^{-t}(A)\triangle A) = 0$ for every $t \in \mathbb{R}$[1].

Definition 4. Let $(X, \mathcal{A}, \mu, \{\Phi^t\}_{t\in\mathbb{R}})$ be a measure preserving flow. We say that Φ^t is *mixing with respect to* μ if for all $f, g \in L^2(X, \mu)$

$$\int_X f \circ \Phi^t \cdot g \,\mathrm{d}\mu \xrightarrow[t\to+\infty]{} \frac{1}{\mu(X)} \int_X f \,\mathrm{d}\mu \int_X g \,\mathrm{d}\mu.$$

Recall the weak convergence in $L^2(X, \mu)$: a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L^2(X, \mu)$ weakly converges to $f \in L^2(X, \mu)$ if for all $g \in L^2(X, \mu)$

$$\int_X f_n g \,\mathrm{d}\mu \xrightarrow[n\to+\infty]{} \int_X f g \,\mathrm{d}\mu.$$

We write $f_n \rightharpoonup f$ in this case. We write $f \circ \Phi^t \rightharpoonup h$ if for every sequence $t_n \to +\infty$, we have $f \circ \Phi^{t_n} \rightharpoonup h$. All the following formulations are equivalent ways to say that Φ^t is mixing:

- $\forall f \in L^2(X, \mu)$, $f \circ \Phi^t \rightharpoonup \frac{1}{\mu(X)} \int_X f \,\mathrm{d}\mu$,
- $\forall A, B \in \mathcal{A}$, $\mu(\Phi^{-t}(A) \cap B) \to \mu(A)\mu(B)/\mu(X)$ when $t \to +\infty$,
- $\forall f \in L^2(X, \mu)$ such that $\int f \mathrm{d}\mu = 0$, $f \circ \Phi^t \rightharpoonup 0$,
- $\forall f \in L^2(X, \mu)$ such that $\int f \mathrm{d}\mu = 0$, all accumulations points of $\{f \circ \Phi^t\}_{t\geq 0}$ are zero.

All equivalencies are easy to see, including the last one if we use the fact that the unit ball of $L^2(X, \mu)$ is sequentially compact in the weak convergence.

It is worth knowing that mixing implies ergodicity. Indeed, suppose A is an invariant set. Then $\mu(\Phi^t(A) \cap A) = \mu(A)$. If Φ^t is mixing, we obtain $\mu(A) = \mu(A)^2/\mu(X)$, so either $\mu(A) = 0$ or $\mu(A) = \mu(X)$. Therefore, Φ^t is ergodic.

The goal of the rest of this section is to prove the following result.

Theorem 5. *Let Γ be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ of finite covolume and let $X = \Gamma\backslash \mathrm{PSL}_2(\mathbb{R})$. The geodesic flow g_t on X is mixing and horocyclic flows h_s^+, h_u^- are ergodic with respect to the Liouville measure μ.*

Until the end of the section, X will denote the quotient $\Gamma\backslash \mathrm{PSL}_2(\mathbb{R})$, where Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ of finite covolume. We begin with a property of X that will be needed later.

Proposition 6. *For all function f in $L^2(X, \mu)$, we have*

$$f \circ g_t \xrightarrow[t\to 0]{L^2} f, \tag{3.4}$$

and similarly for horocyclic flows.

[1] $A\triangle B = (A \setminus B) \cup (B \setminus A)$

Proof. Let D be a domain for Γ as in the definitions of subgroup of finite covolume. Consider the lift $\tilde{f}$ of f in $L^2(X,\mu)$ to $L^2(D,\mu)$. Suppose $\tilde{f}$ is bounded Lipschitz. Then, by dominated convergence we have

$$\lim_{t\to 0}\int_D \left|\tilde{f}\circ g_t - \tilde{f}\right|^2 \mathrm{d}\mu = \int_D \lim_{t\to 0}\left|\tilde{f}\circ g_t - \tilde{f}\right|^2 \mathrm{d}\mu = 0,$$

because $\tilde{f}\circ g_t(x) \to \tilde{f}(x)$ for all x in the interior of D by continuity and $\mu(\partial D) = 0$. Then, we deduce the convergence for f.

Since D has finite measure, bounded Lipschitz functions are dense in $L^2(D,\mu)$. Using approximations we can translate the result to any functions $\tilde{f}$ and f. □

The next two propositions are based on the commuting relations of geodesic and horocyclic flows (Proposition 5).

Proposition 7. *Let f be a function in $L^2(X,\mu)$. If t_n is a sequence of real numbers such that $t_n \to +\infty$ and $f\circ g_{t_n} \rightharpoonup \bar{f} \in L^2(X,\mu)$, then the function $\bar{f}$ is invariant by the flow h_s^+.*

If t_n is a sequence of real numbers such that $t_n \to -\infty$ and $f\circ g_{t_n} \rightharpoonup \bar{f} \in L^2(X,\mu)$, then the function $\bar{f}$ is invariant by the flow h_u^-.

In particular, every g_t-invariant function is invariant by h_s^+ and h_u^-.

Proof. We use Equation 3.1 and we apply that the measure is g_t invariant and the previous proposition to compute the L^2-norm

$$\left\|f\circ g_t\circ h_s^+ - f\circ g_t\right\| = \left\|f\circ h_{se^{-t}}^+\circ g_t - f\circ g_t\right\| = \left\|f\circ h_{se^{-t}}^+ - f\right\| \xrightarrow{t\to+\infty} 0.$$

Hence, the sequence $f\circ g_t\circ h_s^+ - f\circ g_t$ converges weakly to 0. On the other hand, it also converges to $\bar{f}\circ h_s^+ - \bar{f}$. We deduce the first statement by unicity of the limit. The second is done analogously using Equation 3.2. If f is g_t-invariant, then $\bar{f} = f$ for any sequence t_n and we deduce the last statement. □

Proposition 8. *Every h_s^+-invariant function f in $L^2(X,\mu)$ is invariant by the geodesic flow g_t. Every h_u^--invariant function f in $L^2(X,\mu)$ is invariant by the geodesic flow g_t.*

Proof. For any $\varepsilon, s > 0$, we have by Equation 3.3

$$\|f\circ g_{2\log s} - f\| = \left\|f\circ h^+_{\frac{s^{-1}-1}{\varepsilon}}\circ h^-_{\varepsilon}\circ h^+_{\frac{s-1}{\varepsilon}}\circ h^-_{-\varepsilon s^{-1}} - f\right\| =$$

$$\left\|f\circ h^-_{\varepsilon}\circ h^+_{\frac{s-1}{\varepsilon}}\circ h^-_{-\varepsilon s^{-1}} - f\circ h^+_{\frac{s-1}{\varepsilon}}\circ h^-_{-\varepsilon s^{-1}} + f\circ h^-_{-\varepsilon s^{-1}} - f\right\| \leq$$

$$\left\|f\circ h^-_{\varepsilon} - f\right\| + \left\|f\circ h^-_{-\varepsilon s^{-1}} - f\right\| \xrightarrow{\varepsilon\to 0} 0.$$

A similar computation shows the second statement. □

We can now finish the section with the proof of Theorem 5 about the dynamical properties of geodesic and horocyclic flows.

Proof of Theorem 5. For the mixing property of the geodesic flow we will see that, for every square integrable function f with zero integral, all the accumulation points of $\{f\circ g^t\}_{t\geq 0}$ are zero a.e. We already know by Proposition 7 that these accumulation points are h_s^+-invariant.

Let us show that each h_s^+-invariant function $\bar{f}$ is zero a.e. This will prove the mixing of geodesic flow g_t and the ergodicity of the horocyclic flow h_s^+.

By the previous propositions $\bar{f}$ is also g_t and h_u^--invariant. Since the three flows generate the action of $\mathrm{PSL}_2(\mathbb{R})$ by right multiplication, we deduce that

$$\forall A \in \mathrm{PSL}_2(\mathbb{R}),\ \mu\text{-a.e. } x \in X,\ \bar{f}(xA) = \bar{f}(x).$$

Applying Fubini's theorem we get

$$\mu\text{-a.e. } x \in X,\ \mu\text{-a.e. } A \in \mathrm{PSL}_2(\mathbb{R}),\ \bar{f}(xA) = \bar{f}(x).$$

There exists at least one point x_0 in X such that for μ-almost every A in $\mathrm{PSL}_2(\mathbb{R})$, we have $\bar{f}(x_0A) = \bar{f}(x_0)$. Hence we deduce that $\bar{f}$ is constant almost everywhere. Since its integral is zero, we conclude that $\bar{f}$ is zero almost everywhere.

The proof of the ergodicity for h_u^- is done analogously. $\square$

3.2 Geodesic and horocyclic flow on the hyperbolic plane

3.2.1 Unitary tangent bundle and geodesic flow

Recall that geodesic trajectories in the half-plane $\mathbb{H}$ are vertical lines and half-circles with center on the real line. Geodesics themselves are parametrizations of these trajectories by a constant multiple of the arc length parameter. The unitary tangent bundle $T^1\mathbb{H}$ is the set of vectors of $T\mathbb{H}$ with hyperbolic length 1. For each v in $T^1\mathbb{H}$, let γ_v the geodesic satisfying $\gamma_v(0) = \pi(v)$ and $\gamma_v'(0) = v$, where $\pi : T^1\mathbb{H} \to \mathbb{H}$ is the projection. In other words, γ_v is the geodesic trajectory with direction v parametrized by the arc length, say t. For all v in $T^1\mathbb{H}$, the geodesic $\gamma_v(t)$ is defined for all t in $\mathbb{R}$, because both ends of geodesic trajectories have infinite hyperbolic length, which is the reason why the hyperbolic plane is complete. For the same reason, given any two distinct points in the half-plane, there is a unique geodesic which joins them.

The geodesic flow $\{g_t\}_{t\in\mathbb{R}}$ on the unitary tangent bundle is the flow $g_t : T^1\mathbb{H} \to T^1\mathbb{H}$ such that, for all v in $T^1\mathbb{H}$,

$$g_t(v) = \gamma_v'(t).$$

The study of the dynamics of this flow, which has an indubitable geometric interest, goes through the study of its asymptotic behavior. To do this, first we need to look at the space $T^1\mathbb{H}$ and the action of homographies.

Firstly, we need to introduce a distance on the unitary tangent bundle. We can do this as follows: for two unit vectors v_1, v_2 in $T^1\mathbb{H}$ with basepoints $\pi(v_1) = z_1$, $\pi(v_2) = z_2$, let γ be the geodesic joining the two points and define a distance

$$d(v_1, v_2) = d(z_1, z_2) + |\theta_1 - \theta_2|,$$

where $d(x_1, x_2)$ is the hyperbolic distance and θ_1, θ_2 are the angles between v_1, v_2 and the direction of γ at points z_1 and z_2, respectively, as we can see in Figure 3.1. In abstract terms, this distance has a term measuring the distance between their basepoints and a term measuring the angle between vectors, after moving them to the same tangent plane by parallel transport on the geodesic which joins them. Notice that the hyperbolic angles and euclidean angles are the same because the metrics are conformal.

From a more geometrical viewpoint, we can put a natural Riemannian metric on $T^1\mathbb{H}$, called the Sasaki metric. We will define it in general in Chapter 4. The distance obtained from this metric can be seen to be equivalent to the given one.

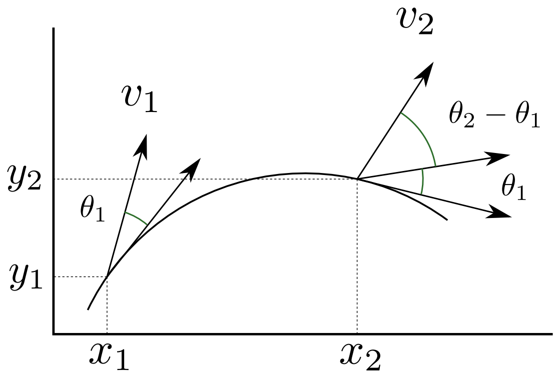

Figure 3.1: Vectors v_1 and v_2 in the hyperbolic plane.

Recall from Section 2.3 that orientation-preserving isometries of $\mathbb{H}$ are homographies associated to $\mathrm{PSL}_2(\mathbb{R})$. The action of these homographies on $\mathbb{H}$ can be extended on $T^1\,\mathbb{H}$ by

$$\begin{array}{ccccc} \mathrm{PSL}_2(\mathbb{R}) & \times & T^1\,\mathbb{H} & \longrightarrow & T^1\,\mathbb{H} \\ \varphi & , & v & \longmapsto & d_{\pi(v)}\varphi(v) = \varphi'(\pi(v)) \cdot v, \end{array}$$

where both points and vectors are thought as complex numbers. From now on, by writing $\varphi(v)$ we will denote the action of $\varphi \in \mathrm{PSL}_2(\mathbb{R})$ on $v \in T^1\,\mathbb{H}$.

The action of $\mathrm{PSL}_2(\mathbb{R})$ on $T^1\,\mathbb{H}$ has better properties than on $\mathbb{H}$. This action is simply transitive, i.e. given any two vectors on $T^1\,\mathbb{H}$ there exists a unique homography in $\mathrm{PSL}_2(\mathbb{R})$ sending the first vector to the second.

Proposition 9. *The action of* $\mathrm{PSL}_2(\mathbb{R})$ *on* $T^1\,\mathbb{H}$ *is simply transitive.*

Proof. Let $v_0 = (\frac{\partial}{\partial y})_i$ in $T_i\,\mathbb{H}$. The element ρ_θ in $\mathrm{PSL}_2(\mathbb{R})$ defined by

$$\rho_\theta(z) = \frac{\cos(\theta/2)z + \sin(\theta/2)}{-\sin(\theta/2)z + \cos(\theta/2)}$$

fixes i in $\mathbb{H}$ and rotates v_0 an angle θ counterclockwise. Let v in $T^1\,\mathbb{H}$ with basepoint $x + iy \in \mathbb{H}$ and let θ be the angle between v and the vertical upward direction. Then, the homography $y \cdot \rho_\theta + x$ in $\mathrm{PSL}_2(\mathbb{R})$ sends v_0 to v.

To see that the action is simple, it is enough to show that if some homography in $\mathrm{PSL}_2(\mathbb{R})$ fixes the vector v_0, it is in fact the identity. Take an homography of the form $\varphi(z) = \frac{az+b}{cz+d}$, $ad - bc = 1$. The condition $\varphi(v_0) = v_0$ implies

$$\varphi(i) = \frac{az+b}{cz+d} = i, \quad d_i\varphi(v_0) = \varphi'(i) \cdot i = \frac{i}{(ci+d)^2} = i.$$

Then, it follows $a = d$, $b = -c$ and $ci + d = \pm 1$, so $c = 0 = b$ and $a = d = \pm 1$ which gives $\varphi = id$ $\square$

Homographies associated to $\mathrm{PSL}_2(\mathbb{R})$ preserve the geometry of $\mathbb{H}$, because they are isometries: lengths and angles of vectors are preserved, geodesics are sent to geodesics, the distance between points and the distance between vectors on $T^1\,\mathbb{H}$ is invariant. We will make use of these properties in the sequel.

3.2.2 Stable and unstable manifolds and horocyclic flows

Next we will introduce two important curves in the unitary tangent bundle which contain information on the asymptotic behavior of the geodesic flow. We can define the *stable manifold* and the *unstable manifold* at $v \in T^1\mathbb{H}$ as the sets

$$W^s(v) = \left\{ u \in T^1\mathbb{H} \,\middle|\, d(g_t(v), g_t(u)) \xrightarrow{t\to+\infty} 0 \right\},$$

$$W^u(v) = \left\{ u \in T^1\mathbb{H} \,\middle|\, d(g_t(v), g_t(u)) \xrightarrow{t\to-\infty} 0 \right\},$$

respectively.

We can explicitly determine these sets. Firstly, we focus in the case $v_0 = (\frac{\partial}{\partial y})_i$ in $T_i\mathbb{H}$. Suppose that u in $T^1\mathbb{H}$ is a vector with basepoint z such that $d(g_t(v_0), g_t(u))$ tends to 0 when t tends to positive infinity. The image of v_0 by the flow is given by $g_t(v_0) = e^t\left(\frac{\partial}{\partial y}\right)_{e^t i}$. We see that

$$d(\pi(g_t(v_0)), \pi(g_t(u))) \geq d(e^t i, \mathrm{Im}(\gamma_u(t))i) = |t - \log \mathrm{Im}(\gamma_u(t))|,$$

so $\mathrm{Im}(\gamma_u(t))$ is unbounded when t approaches the infinity. If the direction u is non-vertical, then the image of γ_u is a semicircle, so its imaginary part is bounded. If u is vertical but downwards, there is also contradiction. We deduce that u is vertical upward, so we can write $u = \mathrm{Im}(z)\left(\frac{\partial}{\partial y}\right)_z$ for some z in $\mathbb{H}$. If $z = x + iy$, then $g_t(u) = e^t y\left(\frac{\partial}{\partial y}\right)_{x+e^t yi}$. By the same argument as before, we obtain a lower bound

$$d(\pi(g_t(v_0)), \pi(g_t(u))) \geq d(e^t i, e^t yi) = |\log y|,$$

so $y = 1$. We have proved that the stable manifold $W^s(v_0)$ is included in the set

$$\left\{ \left(\frac{\partial}{\partial y}\right)_{x+i} \,\middle|\, x \in \mathbb{R} \right\}.$$

Let us prove that, indeed, there is an equality. On one hand, if $u = \left(\frac{\partial}{\partial y}\right)_{x+i}$, the distance between the basepoints is

$$d(\pi(g_t(v_0)), \pi(g_t(u))) = d(e^t i, x + e^t i) \leq \frac{|x|}{e^t} \xrightarrow{t\to+\infty} 0.$$

On the other hand, denoting by θ the angle between the geodesic and the horizontal at the point $e^t i$, the angular part of the distance is

$$|\theta_1 - \theta_2| = 2\theta = 2\arctan\left(\frac{|x|}{2e^t}\right) \xrightarrow{t\to+\infty} 0.$$

See Figure 3.2 for a better understanding. This shows the assertion.

Secondly, using the fact that stable (and unstable) manifolds are preserved by isometries, we can find the stable manifold at any vector. Given a vector v in $T^1\mathbb{H}$, choose the unique homography φ in $\mathrm{PSL}_2(\mathbb{R})$ that sends v_0 to v. Then, the stable manifold at v is $W^s(v) = \varphi(W^s(v_0))$. The projection on $\mathbb{H}$ of this set has to be a line or a circle, because it is the image of a line by a homography. We observe that $\mathrm{PSL}_2(\mathbb{R})$ sends the set $\mathbb{R}\cup\{\infty\}$ to itself. Since φ preserves angles, the set $\pi(W^s(v))$ has to be tangent to the line $\mathbb{R}\cup\{\infty\}$. If v is vertical pointing upward, $\pi(W^s(v))$ has to be the horizontal line (perpendicular to v) that passes through $\pi(v)$. In the other cases, $\pi(W^s(v))$ is the

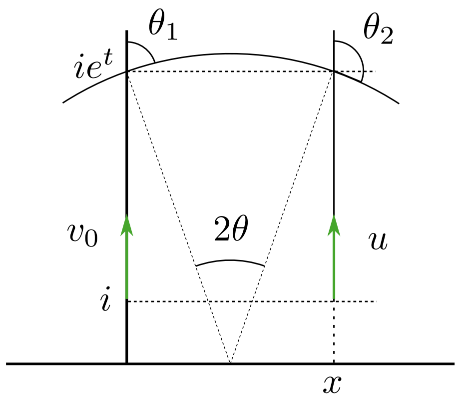

Figure 3.2: Two vertical geodesics starting at the same height.

unique circle tangent to $\mathbb{R}$, passing trough $\pi(v)$, perpendicular to v and v pointing to the interior of the circle. The stable manifold itself is the set of unit vectors with basepoint at $\pi(W^s(v))$, perpendicular to it and pointing inward.

Finally, unstable manifolds can be computed from stable ones using that $W^u(v) = -W^s(-v)$. Usually, the curves $\pi(W^s(v))$ and $\pi(W^u(v))$ are called contracting and expanding horocycles, respectively. In Figure 3.3, we represent a contracting horocycle and its stable manifold.

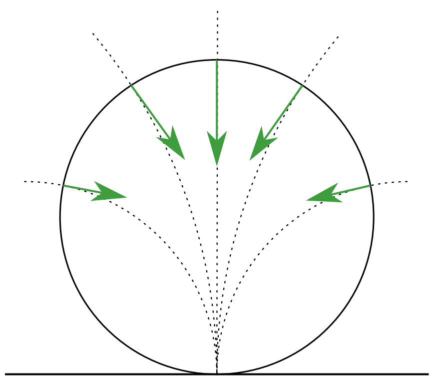

Figure 3.3: Horocycle and stable manifold on the hyperbolic plane.

The *contracting horocyclic flow* $\{h_s^+\}_{s\in\mathbb{R}}$ is a flow on $T^1\,\mathbb{H}$ that parametri- zes the stable manifolds by the length parameter. For all v in $T^1\,\mathbb{H}$, consider the unique φ in $\mathrm{PSL}_2(\mathbb{R})$ such that $\varphi(v_0) = v$. We define

$$h_s^+(v) = \varphi\left(\left(\frac{\partial}{\partial y}\right)_{s+i}\right).$$

Here we are using the fact that homographies respect stable manifolds and lengths, so, since $(\frac{\partial}{\partial y})_{s+i}$ is an arc length parametrization of the stable manifold at v_0, $W^s(v_0)$, its image by φ is so for the stable manifold at v, $W^s(v) = \varphi(W^s(v_0))$.

The *expanding horocyclic flow* $\{h_u^-\}_{u\in\mathbb{R}}$ does the same for unstable manifolds. The homography ρ defined by $\rho(z) = -\frac{1}{z}$ takes the vector v_0 to $-v_0$. Since we have $W^u(v_0) =$

$-W^s(-v_0)$, then we obtain that

$$h_u^-(v) = -\varphi\rho\left(\left(\frac{\partial}{\partial y}\right)_{s+i}\right).$$

parametrizes the unstable manifold at v by arc length.

It is useful to introduce two new sets, called the weak stable manifold $W^{so}(v)$ and the weak unstable manifold $W^{uo}(v)$, defined by

$$W^{so}(v) = \bigcup_{t\in\mathbb{R}} W^s(g_t(v)) = \bigcup_{t\in\mathbb{R}} g_t(W^s(v)),$$

$$W^{uo}(v) = \bigcup_{t\in\mathbb{R}} W^u(g_t(v)) = \bigcup_{t\in\mathbb{R}} g_t(W^u(v)).$$

Thanks to the previous description of the stable and unstable manifolds, that we may call strong to differentiate from the weak manifolds, we see that the weak stable manifold at a vertical upward vector is the set of unit vertical upward vectors with any basepoint. In the other cases, the weak stable manifold is the set of normal inward vectors to circles tangent to the real line with the same point of tangency as the circle $\pi(W^s(v))$.

3.2.3 Correspondence with the algebraic model

In this section, we establish the relation between geodesic and horocyclic flows on the hyperbolic plane and the algebraic model we have described in the first section. The map

$$\begin{array}{cccc} \Psi: & \mathrm{PSL}_2(\mathbb{R}) & \longrightarrow & T^1\,\mathbb{H} \\ & \varphi & \longmapsto & \varphi(v_0) \end{array}$$

is a diffeomorphism. The easiest way to see it is to write the map in coordinates. It is already clear that Ψ is bijective.

A vector v in $T^1\,\mathbb{H}$ has coordinates $(x,y,\theta) \in \mathbb{R}\times(0,+\infty)\times\mathbb{R}/2\pi\mathbb{Z}$, where (x,y) in $\mathbb{H}$ is the basepoint of v and θ is the angle of v with the vertical upward vector $\left(\frac{\partial}{\partial y}\right)_{(x,y)}$. For example, the maps Ψ and Ψ^{-1} in the chart (U_1,ψ_1) of $\mathrm{PSL}_2(\mathbb{R})$ and the previous coordinates on $T^1\,\mathbb{H}$ are

$$\Psi(b,c,d) = \left(\frac{c+b(c^2+d^2)}{d(c^2+d^2)}, \frac{1}{c^2+d^2}, -2\arctan\left(\frac{c}{d}\right)\right),$$

$$\Psi^{-1}(x,y,\theta) = \left(\frac{x\cos\frac{\theta}{2}+y\sin\frac{\theta}{2}}{\sqrt{y}}, -\frac{\sin\frac{\theta}{2}}{\sqrt{y}}, \frac{\cos\frac{\theta}{2}}{\sqrt{y}}\right), \tag{3.5}$$

and we see they are differentiable because d is positive. Similarly, we can compute the expressions of Ψ and its inverse in the other chart and we can see they are also differentiable, so we deduce that Ψ is a diffeomorphism.

Next, we want to know what the geodesic flow on $\mathrm{PSL}_2(\mathbb{R})$ looks like once passed to $T^1\,\mathbb{H}$ by the previous diffeomorphism. If φ is in $\mathrm{PSL}_2(\mathbb{R})$, we compute

$$\begin{aligned}\Psi(g_t(\varphi)) &= \Psi\left(\varphi\left(\begin{smallmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}\end{smallmatrix}\right)\right) = \left(\varphi\left(\begin{smallmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}\end{smallmatrix}\right)\right)(v_0) = \varphi\left(\left(\begin{smallmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}\end{smallmatrix}\right)v_0\right) \\ &= \varphi\left(e^t\left(\frac{\partial}{\partial y}\right)_{e^t i}\right) = \varphi(g_t(v_0)) = g_t(\varphi(v_0)) = g_t(\Psi(\varphi)),\end{aligned}$$

so it turns out that the geodesic flows defined on each side are conjugate by Ψ, that is to say, the following diagram is commutative

$$\begin{array}{ccc} \mathrm{PSL}_2(\mathbb{R}) & \xrightarrow{\Psi} & T^1\,\mathbb{H} \\ \Big\downarrow{\scriptstyle g_t} & & \Big\downarrow{\scriptstyle g_t} \\ \mathrm{PSL}_2(\mathbb{R}) & \xrightarrow{\Psi} & T^1\,\mathbb{H}\,. \end{array}$$

Similar computations show that the two horocyclic flows are also conjugated by Ψ.

It is obvious that the (left) action of $\mathrm{PSL}_2(\mathbb{R})$ on the space $\mathrm{PSL}_2(\mathbb{R})$ and the action of the same group on $T^1\,\mathbb{H}$ are conjugated as well. Then, for any discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$, the diffeomorphism Ψ induces a homeomorphism between the action quotients $\Gamma\backslash\,\mathrm{PSL}_2(\mathbb{R})$ and $T^1\,\mathbb{H}\,/\Gamma$. Geodesic and horocyclic flow are well defined on $T^1\,\mathbb{H}\,/\Gamma$ and conjugated to the ones in $\Gamma\backslash\,\mathrm{PSL}_2(\mathbb{R})$.

On the unitary tangent bundle $T^1\,\mathbb{H}$, consider the pushforward by Ψ of the Liouville measure μ on $\mathrm{PSL}_2(\mathbb{R})$, that we denote also by μ. Automatically, the new measure is invariant by elements of $\mathrm{PSL}_2(\mathbb{R})$ and by the flows g_t, h_s^+ and h_u^-. Then, there is a well defined measure on the quotient $T^1\,\mathbb{H}\,/\Gamma$ invariant by the geodesic and horocyclic flows. The dynamical properties are respected by conjugation, so we deduce the following result.

Theorem 6. *Let Γ be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ with finite covolume. Then the geodesic flow g_t on $T^1\,\mathbb{H}\,/\Gamma$ is mixing and the horocyclic flows h_s^+, h_u^- are ergodic with respect to the Liouville measure μ on $T^1\,\mathbb{H}\,/\Gamma$.*

The hypothesis of covolume finiteness for Γ is equivalent to the finiteness of the volume of the quotient space $T^1\,\mathbb{H}\,/\Gamma$. We now take a closer look at this space.

Proposition 10. *Let Γ be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Then, the subgroup Γ acts totally discontinuously on the unitary tangent bundle $T^1\,\mathbb{H}$. In particular, the quotient $T^1\,\mathbb{H}\,/\Gamma$ is a differentiable manifold.*

Proof. Given any v in $T^1\,\mathbb{H}$ with basepoint x in $\mathbb{H}$, since the action of Γ on $\mathbb{H}$ is properly discontinuous, the orbit Γx is discrete. There exists a ball $B(x,\varepsilon)$ such that $\Gamma x \cap B(x,\varepsilon) = \{x\}$. Denoting $B = B(x,\varepsilon/2)$, we have that $\gamma(B)\cap B \neq \emptyset$ implies that the homography γ is in the stabilizer of x. The lift T^1B of B to the unitary tangent bundle $T^1\,\mathbb{H}$ is an open neighborhood of v such that $\gamma(T^1B)\cap T^1B \neq \emptyset$ implies that γ is in the stabilizer of x. But the latter is finite and we know that the only element of $\mathrm{PSL}_2(\mathbb{R})$ fixing v is the identity, so we can find an open neighborhood W of v in $T^1\,\mathbb{H}$ that satisfies $\gamma(W)\cap W = \emptyset$ for $\gamma \neq id$. □

Recall that for $\mathbb{H}\,/\Gamma$ being a manifold we need a stronger property than Γ being discrete, we need the subgroup Γ to act totally discontinuously. Suppose Γ acts totally discontinuously on $\mathbb{H}$. Then, the projection $p : \mathbb{H} \to \mathbb{H}\,/\Gamma$ is a local isometry between Riemannian manifolds. It can be naturally lifted to the unitary tangent bundles $\tilde{p} : T^1\,\mathbb{H} \to T^1(\mathbb{H}\,/\Gamma)$. The latter is a covering map and the covering group turns out to be Γ, therefore we obtain a natural diffeomorphism $T^1\,\mathbb{H}\,/\Gamma \to T^1(\mathbb{H}\,/\Gamma)$.

In Chapter 4, we will define what the geodesic flow is in a more general situation, a complete Riemannian manifold with negative sectional curvature. The previous diffeomorphism conjugates the geodesic flow on $T^1\,\mathbb{H}\,/\Gamma$ with the geodesic flow defined on the unitary tangent bundle of the manifold $\mathbb{H}\,/\Gamma$.

3.2.4 Liouville measure and Hopf coordinates

Nowadays, the Liouville measure on $T^1\,\mathbb{H}$ is usually defined to be the volume form of the Sasaki metric. However, in this chapter we are not discussing this approach and we just work with its expression in coordinates. We start by computing the expression of the Liouville measure μ in the coordinates (x, y, θ).

Recall that the density of the Liouville measure on $\mathrm{PSL}_2(\mathbb{R})$ is $\mathrm{d}\mu = \frac{1}{d}\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}d$. Then, thanks to Eq. 3.5, we compute

$$\Psi_*\mathrm{d}\mu = \frac{\cos\frac{\theta}{2}}{\sqrt{y}}\left|J\Psi^{-1}\right|\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\theta = \frac{1}{4y^2}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\theta.$$

The factor 1/4 appears as a matter of normalization, the usual Liouville metric does not have the factor. The expression allows to deduce that, in the case that Γ is subgroup of $\mathrm{PSL}_2(\mathbb{R})$ acting totally discontinuously on $\mathbb{H}$, we have that

$$\mu(T^1\,\mathbb{H}/\Gamma) = 2\pi\,\mathrm{Vol}(\mathbb{H}/\Gamma).$$

From this fact and Theorem 6, we deduce the next corollary.

Corollary 1. *Let M be a complete Riemannian surface with curvature -1 and finite volume. Then, the geodesic flow on T^1M is mixing and the horocyclic flows on T^1M are ergodic with respect to the Liouville measure.*

Before ending the section we want to show how μ can be written in a coordinate system that has the weak stable manifold and the unstable manifold as axis.

Let v be a vector in $T^1\,\mathbb{H}$. In a vast majority of cases, the vector v does not point vertically upward, therefore the associated geodesic trajectory is a semicircle perpendicular to the real line at two points. The one that is in the positive direction of v will be denoted by v_+ and the other by v_-, as it is drawn in Figure 3.4.

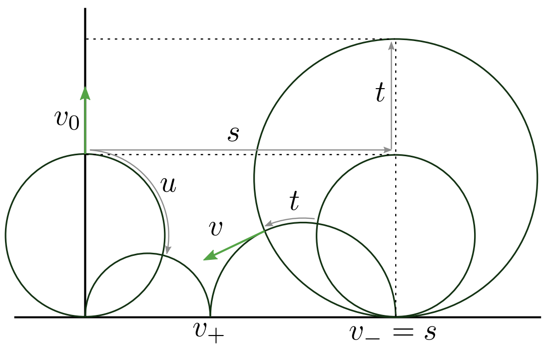

Figure 3.4: Different coordinates of the hyperbolic plane.

The unstable manifold $W^u(v)$ is the set of normal outward vectors to the circle tangent to the real line at v_- and containing the basepoint of v. Then, there is a unique vertical upward vector in $W^u(v)$. Recalling that the weak stable manifold of the vertical upward vector v_0 at i in $\mathbb{H}$ is the set of all unit vertical upward vectors, the previous vector can be written as $g_t(h_s^+(v_0))$ for some s, t in $\mathbb{R}$.

On the other hand, the unstable manifold $W^u(v_0)$ at v_0 is the set of normal outward vectors to the circle containing i and tangent to the real line at 0. There is exactly one vector of $W^u(v_0)$ that lies in the weak stable manifold, namely, the vector on the semicircle containing 0 and v_+. This vector can be written as $h_u^+(v_0)$ for some u in $\mathbb{R}$.

The values (s,t,u) are a system of coordinates defined on a subset of the unitary tangent bundle. The origin of this system is the vector v_0, the (s,t)-axis is the weak stable manifold $W^{so}(v_0)$ and the u-axis is the unstable manifold $W^u(v_0)$.

From Figure 3.4 we see that $s = v_-$. The number t is the distance between the circle tangent to the real line at v_- of euclidean diameter one an the circle tangent at the same point and pasing through the basepoint of v. Then, the values (v_-, v_+, t) are another set of coordinates, called the *Hopf coordinates*. If we apply the homography $z \mapsto -1/z$, the unstable manifold $W^u(v_0)$ transforms to the horizontal line passing through i. Then, the image of $h_u^-(v_0)$ and the image of v_+ by $-1/z$ lie on the same vertical line, so we deduce that $u = -1/v_+$.

We can establish the relation between these coordinates and the classical coordinates (x,y,θ). We can draw the semicircle representing the geodesic trajectory of the vector with coordinates (x,y,θ) and compute the two intersection points with the real axis to obtain the expression of v_- and v_+. Then we can use the homography $z \mapsto 1/(v_- - z)$ that sends v_- to ∞, ∞ to 0, $v_- + i$ to i and the tangent circle at v_- of euclidean diameter 1 to the horizontal line passing through i. Then, t is the distance between this line and the image of the point $x + yi$, which is a very simple computation. By this method we obtain the following relations:

$$v_- = x + y\tan(\theta/2),$$
$$v_+ = x - \frac{y}{\tan(\theta/2)},$$
$$t = \log y - 2\log|\cos(\theta/2)|.$$

The Hopf coordinates (v_-, v_+, t) are well defined from the whole space $T^1\mathbb{H}$ to the set $(\bar{\mathbb{R}} \times \bar{\mathbb{R}} - \Delta) \times \mathbb{R}$, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and Δ is the diagonal of $\bar{\mathbb{R}} \times \bar{\mathbb{R}}$. Computing the Jacobian of the transformations, we can write the expression of the Liouville measure in the different sets of coordinates, obtaining

$$\mathrm{d}\mu = \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\theta}{y^2} = 2\frac{\mathrm{d}v_-\mathrm{d}v_+\mathrm{d}t}{(v_+ - v_-)^2} = 2\frac{\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u}{(su+1)^2}.$$

Chapter 4

Geodesic flow in negative curvature

In this chapter we will study the geodesic flow in manifolds of varible negative curvature. The main goal will be to prove the ergodic property of the flow when the manifold is compact using the so-called Hopf argument. To understand the behavior of the flow it is important to study the geometry of negative curvature. As we will see, the geodesic flow is related to Jacobi fields, which at the same time are linked to the curvature of the manifold. We will also need to formalize the notion of horospheres, which play the role of horocyclic flow in higher dimension, and stable and unstable manifolds. All this will be done in Section 4.1.

In Section 4.2, we will explain how the proof of the ergodicity proceeds and what ingredients we need. It will remain to prove a rather technical property called absolute continuity. In Section 4.3, we will look at a class of flows that generalize the geodesic flow. They satisfy a property that will be needed in the proof of absolute continuity. Finally, we will be able to prove absolute continuity and, thus, the ergodicity of geodesic flow in Section 4.4.

This chapter is inspired in the notes [Bal95] of W. Ballmann, that include an appendix dedicated to the ergodic property of the geodesic flow.

4.1 Geometry in negative curvature

4.1.1 Geodesic flow

Our goal in this preliminary section is to define the geodesic flow in a general manifold with a Riemannian metric. This flow is defined on the tangent bundle of the manifold and it can be restricted to the unitary tangent bundle, that is where we will work. Certain properties of the geodesic flow are reflected by its tangent map defined in the double tangent bundle. Next, we provide a nice description of this space.

Let M be a Riemannian manifold. We assume that the reader is already familiar with the structure as manifold of the tangent bundle TM of M. Let $\pi : TM \to M$ denote the projection. We consider the bundle

$$\mathcal{E} = \pi^*(TM) \oplus \pi^*(TM)$$

of the tangent bundle TM. The fiber at the point v in TM is the space $T_xM \oplus T_xM$, where $x = \pi(v)$.

We can also consider the tangent bundle TTM of the manifold TM, know as the double tangent bundle. For each Z in TTM, there exists a curve $V : (-\delta, \delta) \to TM$ such that $\dot{V}(0) = Z$. Set $c = \pi \circ V$ and define a map $\mathcal{I} : TTM \to \mathcal{E}$ by

$$\mathcal{I}(Z) = \left(\dot{c}(0), \frac{DV}{dt}(0)\right),$$

where $\frac{DV}{dt}$ is the covariant derivative of the field V at 0 in the direction $\dot{c}(0)$.

Recall that a system of coordinates in the manifold M gives a compatible system of coordinates of TM. If we write the map $\mathcal{I}$ in these coordinate charts, it is not difficult to see the following result.

Lemma 2. *The map $\mathcal{I} : TTM \to \mathcal{E}$ is an isomorphism of vector bundles.*

From now on, we will identify the double tangent bundle TTM to the vector bundle $\mathcal{E}$ with no further mention.

To define the geodesic flow we will additionally suppose that the manifold M is complete. Given v in TM, let γ_v denote the unique geodesic satisfying $\dot{\gamma_v}(0) = v$. The geodesic flow $g_t : TM \to TM$ on the tangent bundle is then defined by

$$g_t(v) = \dot{\gamma_v}(t), \quad \forall t \in \mathbb{R}\,.$$

Jacobi fields are very useful to describe the geodesic flow. Let us make a reminder of these fields. Let $\gamma : [0, a] \to M$ be a geodesic. We say that a vector field $J : [0, a] \to TM$ along γ, i. e. $\pi \circ J = \gamma$, is a Jacobi field if it satisfies the Jacobi equation

$$J''(t) + R(\dot{\gamma}(t), J(t))\dot{\gamma}(t) = 0, \quad \forall t \in [0, a], \tag{4.1}$$

where J'' stands for the second covariant derivative of J in the direction of the geodesic $\dot{\gamma}(t)$ and R is the Riemann curvature tensor.

The Jacobi equation 4.1 is a linear differential equation of second order, so given initial conditions $J(0)$ and $J'(0)$ there exists a unique solution of the equation. The following result is easy to prove with some computations, but it is out of the topic of this dissertation. It can be found in Chapter 5 of [dC92].

Lemma 3. *Let $V : [0, a] \to TM$ be a field with $V(0) = v$. Then the field J defined by*

$$J(s) = \left.\frac{\partial}{\partial t}\right|_{t=0} \gamma_{V(t)}(s)$$

is a Jacobi field on the geodesic γ_v.

Let v in TM and let (X, Y) be a vector in T_vTM. Consider a smooth curve V on TM such that $\dot{c}(0) = X$ and $\frac{DV}{dt}(0) = Y$. We consider the Jacobi field

$$J(s) = \left.\frac{\partial}{\partial t}\right|_{t=0} \gamma_{V(t)}(s)$$

on the geodesic γ_v. The tangent map of the geodesic flow is

$$d_v g_s(X, Y) = \left.\frac{d}{dt}\right|_{t=0} g_s(V(t)).$$

If we denote $W(t) = g_s(V(t)) = \dot{\gamma}_{V(t)}(s)$, then $\pi \circ W(t) = \gamma_{V(t)}(s)$ and we obtain

$$d_v g_s(X, Y) = \left((\pi \circ W)'(0), \frac{DW}{dt}(0)\right) = \left(\left.\frac{\partial}{\partial t}\right|_{t=0} \gamma_{V(t)}(s), \left.\frac{D}{\partial t}\right|_{t=0} \dot{\gamma}_{V(t)}(s)\right)$$

$$= \left(J(s), \frac{D}{\partial t}\bigg|_{t=0} \frac{\partial}{\partial s}\gamma_{V(t)}(s) \right) = \left(J(s), \frac{D}{\partial s}\frac{\partial}{\partial t}\bigg|_{t=0} \gamma_{V(t)}(s) \right) = (J(s), J'(s)).$$

The Jacobi field J is determined by $J(0) = X$ and $J'(0) = Y$. We deduce the expression of the differential of the geodesic flow.

Proposition 11. *Let $s \in \mathbb{R}$ and $v \in TM$. The tangent map of the geodesic flow $g_t : TM \to TM$ acts by*

$$d_v g_s(X, Y) = (J(s), J'(s)),$$

where J is the unique Jacobi field such that $J(0) = X$ and $J'(0) = Y$.

Let us now talk about the Sasaki metric on the unit tangent bundle that we have already mentioned. Using the decomposition of the double tangent bundle, for all $(X_1, Y_1), (X_2, Y_2)$ in TTM, we define the metric

$$\langle (X_1, Y_1), (X_2, Y_2) \rangle = \langle X_1, X_2 \rangle + \langle Y_1, Y_2 \rangle, \tag{4.2}$$

where the right products are done using the Riemannian metric of M. This endows the tangent bundle with a structure of Riemannian manifold.

We introduce a differential 1-form on TM by

$$\alpha_v((X, Y)) = \langle v, X \rangle,$$

for v in TM, and a differential 2-form

$$\omega((X_1, Y_1), (X_2, Y_2)) = \langle X_2, Y_1 \rangle - \langle X_1, Y_2 \rangle. \tag{4.3}$$

Working in coordinates we can see that $d\alpha = \omega$. Next, we observe that ω is invariant by the geodesic flow. Let J_1, J_2 two Jacobi fields such that $J_i(0) = X_i$ and $J_i'(0) = Y_i$ for $i = 1, 2$. Then, using the Jacobi equation and the symmetries of the curvature tensor we have

$$\frac{d}{dt}\omega(d_v g_t(X_1, Y_1), d_v g_t(X_2, Y_2)) = \frac{d}{dt}\left(\langle J_2(t), J_1'(t) \rangle - \langle J_1(t), J_2'(t) \rangle \right)$$

$$= \langle J_2, J_1'' \rangle - \langle J_1, J_2'' \rangle = -\langle R(\dot\gamma_v, J_1)\dot\gamma_v, J_2 \rangle + \langle R(\dot\gamma_v, J_2)\dot\gamma_v, J_1 \rangle = 0.$$

We claim that ω^n is the volume form of the Sasaki metric. This follows from the fact that ω^n evaluated at an orthonormal basis is ± 1. In effect, for a fixed v in TM take an orthonormal basis $e_1, \ldots, e_n$ of $T_{\pi(v)}M$. Then the vectors $F_1, G_1, \ldots, F_n, G_n$, where $F_i = (0, e_i) \in T_vTM$ and $G_i = (e_i, 0) \in T_vTM$, form an orthonormal basis of the Sasaki metric 4.2. Using 4.3 we see that the form ω_v in the dual basis is written

$$\omega_v = \sum_{i=1}^{n} F_i^* \wedge G_i^*.$$

It follows that

$$\omega_v^n = F_1^* \wedge G_1^* \wedge \cdots \wedge F_n^* \wedge G_n^*$$

and we conclude that ω^n is the volume form. The measure associated to this volume form ω^n is called the *Liouville measure* and it is invariant by the geodesic flow, because ω is.

Let us turn our attention to the unitary tangent bundle T^1M, which is a submanifold of TM of one dimension less. An important remark is that the normal direction of this submanifold at the point v in T^1M is given by $(0,v)$ in T_vTM. Consequently, the tangent spaces of this submanifold can be described as

$$T_vT^1M = \{(X,Y) \in T_vTM \,|\, Y \perp v\}.$$

The geodesic flow sends the unitary tangent bundle T^1M to itself. The differential forms α and ω can be restricted to T^1M, we also denote them α and ω. The fact that we are in the unitary tangent bundle implies that α is also invariant by the geodesic flow. For all v in T^1M and (X,Y) in T_vT^1M, take the Jacobi field J such that $J(0) = X$ and $J'(0) = Y$ and compute

$$\begin{aligned}\frac{d}{dt}\alpha_{g_t(v)}(d_v g_t(X,Y)) &= \frac{d}{dt}\alpha_{g_t(v)}(J(t), J'(t)) = \frac{d}{dt}\langle \dot\gamma_v(t), J(t)\rangle \\ &= \langle \dot\gamma_v(t), J'(t)\rangle = \langle \dot\gamma_v(0), J'(0)\rangle = \langle v, Y\rangle = 0.\end{aligned}$$

The unitary tangent bundle inherits a Riemannian structure, whose volume form at point v in TM is

$$\iota_{(0,v)}\omega_v^n = n\, \iota_{(0,v)}\omega_v \wedge \omega_v^{n-1},$$

but for all (X,Y) in T_vT^1M, we have

$$\iota_{(0,v)}\omega_v(X,Y) = \omega_v((0,v),(X,Y)) = \langle v, X\rangle = \alpha_v(X,Y),$$

so we deduce that the volume form of the unitary tangent bundle is $\iota_{(0,v)}\omega^n = n\alpha \wedge \omega^{n-1}$. The measure defined by this volume form is also called the Liouville measure. We have proved the following result.

Theorem 7. *Let M be a complete Riemannian manifold. The Liouville measure on the unitary tangent bundle T^1M is invariant by the geodesic flow $g_t : T^1M \to T^1M$.*

Let us show what the Sasaki metric looks like in the case of the hyperbolic plane. We use the coordinates (x,y) in $\mathbb{H}$. Then the tangent bundle $T\,\mathbb{H}$ has a compatible set of coordinates $(\bar x, \bar y, \xi, \eta)$, where for all v in $T\,\mathbb{H}$, $\bar x(v) = x(\pi(v))$, $\bar y(v) = y(\pi(v))$ and

$$v = \xi \left(\frac{\partial}{\partial x}\right)_{\pi(v)} + \eta \left(\frac{\partial}{\partial y}\right)_{\pi(v)}.$$

The unitary tangent bundle $T^1\,\mathbb{H}$ is equipped with the coordinates $(\bar x, \bar y, \theta)$ as in Chapter 3.

First of all we need to compute the expression of the basis of the tangent space of $T\,\mathbb{H}$ via the identification $\mathcal{I}$. We obtain

$$\begin{aligned}\mathcal{I}\left(\frac{\partial}{\partial \bar x}\right) &= \left(\left(\frac{\partial}{\partial x}\right), \frac{\xi}{y}\left(\frac{\partial}{\partial y}\right) - \frac{\eta}{y}\left(\frac{\partial}{\partial x}\right)\right), \\ \mathcal{I}\left(\frac{\partial}{\partial \bar y}\right) &= \left(\left(\frac{\partial}{\partial y}\right), -\frac{\xi}{y}\left(\frac{\partial}{\partial x}\right) - \frac{\eta}{y}\left(\frac{\partial}{\partial y}\right)\right), \\ \mathcal{I}\left(\frac{\partial}{\partial \xi}\right) &= \left(0, \left(\frac{\partial}{\partial x}\right)\right), \\ \mathcal{I}\left(\frac{\partial}{\partial \eta}\right) &= \left(0, \left(\frac{\partial}{\partial y}\right)\right).\end{aligned}$$

To obtain the expressions in the coordinates of $T^1\,\mathbb{H}$ we set $\xi = -r\sin\theta$, $\eta = r\cos\theta$ and we let $r = 1$. Using the definitions of the Sasaki metric we compute its matrix in the basis of $(\bar{x}, \bar{y}, \theta)$,

$$\begin{pmatrix} \frac{2}{y^2} & 0 & \frac{1}{y} \\ 0 & \frac{2}{y^2} & 0 \\ \frac{1}{y} & 0 & 1 \end{pmatrix}.$$

It follows that the volume form in these coordinates is

$$\frac{\sqrt{2}}{y^2}\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\,\mathrm{d}\theta$$

which coincides with the Liouville measure density we used in Chapter 3, up to a constant.

4.1.2 Jacobi fields in nonpositive curvature

In the previous section, we have seen that there is a relation between the geodesic flow and Jacobi fields. In the present one, we will study some properties of the Jacobi fields in nonpositive sectional curvature that will be needed in the future.

Let us introduce a particular type of Jacobi field.

Definition 5. Let M be a Riemannian manifold of nonpositive curvature. We say that a Jacobi field J along a unit speed geodesic $\gamma : \mathbb{R} \to M$ is *stable* if its norm is bounded for nonnegative time, i.e there exists a constant $C > 0$ such that foll all $t \geq 0$ we have $\|J(t)\| \leq C$.

We will need the following properties which are proved in [Bal95, IV.2.8].

Proposition 12. *Let M be a Riemannian manifold of nonpositive curvature and $\gamma : \mathbb{R} \to M$ a unit speed geodesic. Set $p = \gamma(0)$.*

(i) For all vector X in T_pM, there exists a unique stable Jacobi field J_X along γ such that $J_X(0) = X$.

(ii) Let $\{\gamma_n\}_{n\in\mathbb{N}}$ a sequence of unit speed geodesics converging to γ. Let J_n be a Jacobi field along γ_n, for all natural n. Suppose that $J_n(0) \to X$ and that there is a constant $C > 0$ and a sequence of real numbers converging to infinity $t_n \to +\infty$ such that $\|J_n(t_n)\| \leq C$ for all n. Then we have $J_n \to J_X$ and $J'_n \to J'_X$

The main goal of this section are the following estimates of stable Jacobi fields, which will allow to control the growth of the geodesic flow.

Proposition 13. *Let M be a Riemannian manifold of nonpositive curvature, $\gamma : \mathbb{R} \to M$ a unit speed geodesic and J a stable Jacobi field along γ perpendicular to $\dot{\gamma}$.*

(i) If the curvature of M along the geodesic γ is bounded from above by $-a^2$, where $a \geq 0$, then we have

$$\|J(t)\| \leq \|J(0)\|\, e^{-at} \quad \textit{and} \quad \left\|J'(t)\right\| \geq a\,\|J(t)\| \quad \forall t \geq 0.$$

(ii) If the curvature of M along the geodesic γ is bounded from below by $-b^2$, where $b \geq 0$, then we have

$$\|J(t)\| \geq \|J(0)\|\, e^{-bt} \quad \textit{and} \quad \left\|J'(t)\right\| \leq b\,\|J(t)\| \quad \forall t \geq 0.$$

We will show how the proof goes in the case of curvature bounded from above. The proof for the second case does not follow the same method. We start with a lemma that estimates the second derivative of the norm of a Jacobi field.

Lemma 4. *Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic. Suppose that the curvature of M along γ is bounded above by a constant k. If J is a Jacobi field along γ perpendicular to $\dot\gamma$, then for all t where the field J does not vanish we have*

$$\|J\|''(t) \geq -k\|J\|(t).$$

Proof. We compute the second derivative

$$\|J\|'' = \left(\frac{\langle J', J\rangle}{\|J\|}\right)' = \frac{1}{\|J\|^2}\left(\langle J'', J\rangle\|J\| + \langle J', J'\rangle\|J\| - \frac{\langle J', J\rangle^2}{\|J\|}\right)$$

$$= \frac{1}{\|J\|^3}\left(\langle -R(\dot\gamma, J)\dot\gamma, J\rangle\|J\|^2 + \|J'\|^2\|J\|^2 - \langle J', J\rangle^2\right) \geq -\frac{\langle R(\dot\gamma, J)\dot\gamma, J\rangle}{\|J\|}.$$

The sectional curvature at the plane spanned by $\dot\gamma$ and J is

$$\frac{\langle R(\dot\gamma, J)\dot\gamma, J\rangle}{\|\dot\gamma\|^2\|J\|^2 - \langle\dot\gamma, J\rangle^2} = \frac{\langle R(\dot\gamma, J)\dot\gamma, J\rangle}{\|J\|^2}$$

and is smaller than k. Hence, we obtain the lower bound of the second derivative. $\square$

The same computation shows that in the case of nonpositive curvature the function $\|J\|$ is convex, so it cannot have more than one zero if the Jacobi field is nonzero. If J is a Jacobi field such that $J(0) = 0$, then the consequence of the lemma is valid for all t different from 0. In addition, we observe that the tangent direction of a Jacobi field is

$$\langle\dot\gamma(t), J(t)\rangle = \langle\dot\gamma(0), J'(0)\rangle t + \langle\dot\gamma(0), J(0)\rangle,$$

so if $J(0) = 0$ the fact that J is perpendicular to $\dot\gamma$ means that $J'(0)$ and $\dot\gamma(0)$ are perpendicular.

Proposition 14. *Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic. Suppose that the curvature of M along γ is bounded above by a constant $k = -a^2 < 0$. If J is a Jacobi field along γ with $J(0) = 0$, $J'(0)\perp\dot\gamma(0)$, then for $t > 0$ we have*

$$\|J\|'(t) \geq a\coth(at)\|J(t)\|.$$

Proof. By Lemma 4, the quantity

$$\left(\|J\|'(t)\sinh(at) - a\|J\|(t)\cosh(at)\right)' = \left(\|J\|''(t) - a^2\|J\|(t)\right)\sinh(at).$$

is nonnegative for positive time t. Then, for $t > 0$,

$$\|J\|'(t)\sinh(at) - a\|J\|(t)\cosh(at) \geq 0, \tag{4.4}$$

and the statement follows. $\square$

With these ingredients we can prove the estimates for the case of bounded above curvature.

Proof of Proposition 13 (i). For each $n \geq 1$ consider the Jacobi field J_n along γ such that $J_n(0) = J(0)$ and $J_n(n) = 0$. Notice that J is perpendicular to $\dot{\gamma}$ at least at two points, so it has to be perpendicular everywhere. Proposition 14 applied to the field $\tilde{J}_n$ defined by $\tilde{J}_n(t) = J_n(n-t)$ gives

$$\frac{\|\tilde{J}_n\|'}{\|\tilde{J}_n\|}(t) \geq a\coth(at). \tag{4.5}$$

Integrating both sides of 4.5 we obtain between $n-t$ and n, and then taking the exponential, we obtain for $t < n$

$$\frac{\|\tilde{J}_n(n)\|}{\|\tilde{J}_n(n-t)\|}(t) \geq \frac{\sinh(an)}{\sinh(a(n-t))}. \tag{4.6}$$

Equations 4.5 and 4.6 in terms of J_n and J are

$$\|J_n\|'(t) \leq -a\coth(a(n-t))\,\|J_n(t)\|,$$

$$\frac{\|J_n(t)\|}{\|J(0)\|} \leq \frac{\sinh(a(n-t))}{\sinh(an)}.$$

Proposition 12 implies $J_n \to J$ and $J_n' \to J'$ when n tends to infinity. The limit of the last expression gives the first formula in the statement. For the other we notice that

$$\left|\|J_n\|'\right| = \frac{|\langle J_n, J_n'\rangle|}{\|J_n\|} \leq \|J_n'\|$$

and use the previous bounds.

□

To finish the section we state an estimate of the distance between two geodesic flow orbits that will be needed later [Bal95, IV.2.10]. We let d_1 denote the distance on the unitary tangent bundle.

Proposition 15. *Let M be a Hadamard manifold. Suppose that the sectional curvature is pinched between two constants $-b^2 \leq -a^2 < 0$. Then for every constant $D > 0$ there exist constants $C, T \geq 1$ such that*

$$d_1(g_t(v), g_t(w)) \leq Ce^{-at}d_1(v,w), \quad 0 \leq t \leq R,$$

where v and w are inward unit vectors to a geodesic sphere of radius $R \geq T$ in M with basepoints x and y at distance less than D.

4.1.3 Horospheres

We continue the study of the geodesic flow, this time we will look at some objects called horospheres that are intimately related to the stable and unstable manifolds. To define them we will first talk about Busemann functions. We will work on a *Hadamard manifold M*, which is a simply connected, complete Riemannian manifold with nonpositive curvature.

Definition 6. A ray $\sigma : [0, +\infty) \to M$ is a unit speed minimizing geodesic.

The word minimizing means that, for all t, t' in $[0, +\infty)$, the distance between $\gamma(t)$ and $\gamma(t')$ is $|t - t'|$. A geodesic is always locally minimizing, but here we are requiring that it holds globally.

Definition 7. Two rays σ_1, σ_2 are asymptotic if $d(\sigma_1(t), \sigma_2(t))$ is bounded uniformly in $t \geq 0$.

To be asymptotic is an equivalent relation over the set of all rays in M. We define the *closure of* M as the set of equivalence classes by this relation and it will be denoted by $M(\infty)$. If ξ is an equivalence class of asymptotic rays and σ is a ray in ξ, we will write $\sigma(\infty) = \xi$.

Let $\sigma_n : [0, l_n]$ be sequence of unit speed geodesic segments on M of length tending to infinity, i. e. $l_n \to +\infty$. We say that the sequence σ_n converges to a ray σ of M if for all real numbers $\varepsilon > 0$ and $R > 0$, there is a natural n_0 such that , for all $n \geq n_0$, we have $l_n > R$ and

$$d(\sigma_n(t), \sigma(t)) < \varepsilon$$

for t in $[0, R]$.

To continue our discussion we need the next fact from nonpositive curvature geometry (see [Bal95, I.5.4]).

Lemma 5. *Let M be a Hadamard manifold and let $\sigma_1, \sigma_2 : I \to M$ be two unit speed geodesics on M. Then the function*

$$d(\sigma_1(t), \sigma_2(t))$$

of t is convex.

Thanks to this lemma we can prove the following fact about rays.

Proposition 16. *For each x in M and each ξ in $M(\infty)$, there is a unique ray $\sigma_{x,\xi} : [0, +\infty) \to M$ such that $\sigma_{x,\xi}(0) = x$ and $\sigma_{x,\xi}(\infty) = \xi$.*

Sketch of the proof. The unicity follows from the fact that if σ_1, σ_2 are two asymptotic rays starting at the same point, then the function

$$d(\sigma_1(t), \sigma_2(t))$$

of $t \in [0, +\infty)$ is 0 at the point 0, is bounded and it is convex by Lemma 5, so it is 0 everywhere.

Let σ be a ray representing ξ. We consider the parametric family of geodesic segments σ_T connecting the point x to the point $\sigma(T)$, where $T > 0$. Using again the properties of the manifolds with nonpositive curvature (see [Bal95, II.2.1]), we prove that: given two real numbers $\varepsilon > 0$ and $R > 0$, there exists $T_0 > 0$ such that $T, S \geq T_0$ implies

$$d(\sigma_T(t), \sigma_S(t)) < \varepsilon$$

for all t in $[0, R]$.

We deduce that σ_T converges to some ray $\sigma_{x,\xi}$ starting at x and asymptotic to σ, or equivalently $\sigma_{x,\xi}(\infty) = \xi$. $\square$

Let us introduce an essential notion related to the horospheres.

Definition 8. Let x be a point in M and ξ a point of the closure $M(\infty)$. The *Busemann function* $b_{x,\xi} : M \to \mathbb{R}$ at ξ based at x is defined by

$$b_{x,\xi}(z) = \lim_{t \to +\infty} \left(d(\sigma(t), z) - t \right),$$

for all z in M, where $\sigma = \sigma_{x,\xi}$ is the unique ray such that $\sigma_{x,\xi}(0) = x$ and $\sigma_{x,\xi}(\infty) = \xi$.

The limit in the definition exists because the function $h(t) = d(\sigma(t), z) - t$ is bounded from below and decreasing. In effect, we have

$$t = d(\sigma(0), \sigma(t)) \leq d(\sigma(0), z) + d(z, \sigma(t)),$$

so $h(t) \geq -d(x, z)$ and if $t' > t$, we also have

$$d(\sigma(t'), z) \leq d(\sigma(t'), \sigma(t)) + d(\sigma(t), z) = t' - t + d(\sigma(t), z),$$

so $h(t') \leq h(t)$.

The Busemann function is continuous, in fact, a simple computation shows that

$$|b_{x,\xi}(y) - b_{x,\xi}(z)| \leq d(y, z).$$

We want to see that Busemann functions are approximated by some simpler functions. Let $b_{x,y} : M \to \mathbb{R}$ denote the function defined by

$$b_{x,y}(z) = d(y, z) - d(y, x).$$

Definition 9. We say that a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ converges to a point ξ of the closure $M(\infty)$ if the distance $d(x_0, x_n) \to +\infty$ and the minimizing geodesic segments σ_{x_0,x_n} connecting the point x_0 to the point x_n tend to the ray $\sigma_{x_0,\xi}$.

We remark that the previous definition does not depend on the point x_0 in the following sense: if $\{x_n\}$ is a sequence converging to ξ in $M(\infty)$ and y is a point in M, then the geodesic segments σ_{y,x_n} converge to the geodesic ray $\sigma_{y,\xi}$. It can be seen with a similar argument to the one used in the proof of Proposition 16.

The reason for introducing these last notions is the following result.

Proposition 17. *Let $\{x_n\}$ be a sequence converging to ξ in $M(\infty)$ and x be a point in M. Then we have*

$$\lim_{n\to+\infty} b_{x,x_n} = b_{x,\xi}.$$

Proof. The fact that $\sigma_{x,x_n} \to \sigma_{x,\xi}$ implies that $d(\sigma_{x,\xi}(d(x, x_n)), x_n) \to 0$ as $n \to +\infty$. In addition, we have $d(x, x_n) \to +\infty$, and therefore, for all z in M,

$$\begin{aligned} b_{x,\xi}(z) &= \lim_{t\to+\infty} (d(\sigma_{x,\xi}(t), z) - t) = \lim_{n\to+\infty} (d(\sigma_{x,\xi}(d(x, x_n)), z) - d(x, x_n)) \\ &= \lim_{n\to+\infty} (d(x_n, z) - d(x_n, x)) = \lim_{n\to+\infty} b_{x,x_n}(z). \end{aligned}$$

□

In the previous proposition we have proved that Busemann functions are a pointwise limit. In fact, it can be seen with some geometry of nonpositive curvature that the convergence actually is uniform on compact sets [Bal95, II]. This fact will be used in this dissertation without proof.

A *horosphere* is a level set of a Busemann function. More precisely, for some x in M and ξ in $M(\infty)$, the level set

$$\{z \in M \mid b_{x,\xi}(z) = 0\}$$

is called the *horosphere centered at ξ passing through x*. Observe that $b_{x,\xi}(x) = 0$ so the point x is contained in a horosphere passing through x.

It is clear that, for all points x, y, z, w in M, the equality

$$b_{x,w}(z) - b_{y,w}(z) = b_{x,w}(y)$$

holds. By applying it to the points x, y, z, w_n, where w_n is a sequence of points converging to ξ in $M(\infty)$, and passing to the limit, we obtain the equality

$$b_{x,\xi}(z) - b_{y,\xi}(z) = b_{x,\xi}(y).$$

This implies the equality between horospheres

$$\{z \in M \mid b_{y,\xi}(z) = R\} = \{z \in M \mid b_{x,\xi}(z) = R + b_{x,\xi}(y)\},$$

where R is a real number, so the set of horospheres given by the function $b_{x,\xi}$ is the same as the set of the ones given by $b_{y,\xi}$, whatever the points x and y in M are.

Let x be a point of M and $\{x_n\}$ be a sequence of points of M converging to ξ in $M(\infty)$. The level set $b_{x,x_n}^{-1}(0)$ is the sphere centered at x_n passing through x. Since we have $b_{x,x_n} \to b_{x,\xi}$, we can say that the horosphere centered at ξ passing through x is the limit of spheres passing through x when their centers tend to ξ and the convergence is uniform on compact sets.

Now we introduce some subspaces of the unitary tangent bundle T^1M. The *(strong) stable manifold at v* in T^1M is the set

$$W^s(v) = \{w \in T^1M \mid d(\gamma_v(t), \gamma_w(t)) \xrightarrow{t\to+\infty} 0\}.$$

First of all, we also remark that

$$g_t(W^s(v)) = W^s(g_t(v)) \quad \forall t \in \mathbb{R}\,.$$

In the next proposition we reveal the relation between stable manifolds and horospheres.

Proposition 18. *Let v be a vector in T^1M with basepoint x and let ξ denote the asymptotic class of the ray associated to v. Then the stable manifold at v is the set of vectors with associated ray in ξ and starting point on the horosphere centered at ξ passing through x, i. e.*

$$W^s(v) = \{\dot\sigma_{y,\xi}(0) \mid y \in M,\, b_{x,\xi}(y) = 0\},$$

Proof. Let w be a vector in $W^s(v)$ with basepoint y in M. The ray $\sigma_{x,\xi}$ is the restriction of the geodesic γ_v starting at v to nonnegative time. Moreover, it is clear that γ_v and γ_w are asymptotic, so $\sigma_{y,\xi}$ is the restriction of the geodesic γ_w to nonnegative time and it follows that $w = \dot\sigma_{y,\xi}(0)$. Let us show that $b_{x,\xi}(y) = 0$. We have

$$|b_{x,\xi}(y)| = \lim_{t\to+\infty} |d(\sigma_{x,\xi}(t), y) - t| = \lim_{t\to+\infty} |d(\sigma_{x,\xi}(t), y) - d(\sigma_{y,\xi}(t), y)| \leq$$

$$\lim_{t\to+\infty} d(\sigma_{x,\xi}(t), \sigma_{y,\xi}(t)) = \lim_{t\to+\infty} d(\gamma_v(t), \gamma_w(t)) = 0,$$

so y is in the horosphere centered at ξ passing trough x.

Conversely, let y be a point in the horosphere centered at ξ passing through x and we must show that

$$d(\sigma_{x,\xi}(t), \sigma_{y,\xi}(t)) \xrightarrow{t\to+\infty} 0.$$

For any sequence of real numbers $t_n \to +\infty$, consider the sequence of points $x_n = \sigma_{x,\xi}(t_n)$. This sequence converges to ξ in the closure $M(\infty)$. Then we have

$$d(\sigma_{x,\xi}(t_n), \sigma_{y,\xi}(t_n)) = d(x_n, \sigma_{y,\xi}(d(x, x_n))) \leq d(x_n, \sigma_{y,\xi}(d(y, x_n)))+$$

$$\begin{aligned} d(\sigma_{y,\xi}(d(y,x_n)),\sigma_{y,\xi}(d(x,x_n))) &= d(x_n,\sigma_{y,\xi}(d(y,x_n))) + |d(x_n,y) - d(x_n,x)| \\ &= d(x_n,\sigma_{y,\xi}(d(y,x_n))) + |b_{x,x_n}(y)| \end{aligned}$$

and both terms tend to zero, the first because $\sigma_{y,x_n} \to \sigma_{y,\xi}$ and the second because $b_{x,x_n}(y) \to b_{x,\xi}(y) = 0$. □

We define the *weak stable manifold at* v as the set

$$W^{so}(v) = \{w \in T^1M \mid \gamma_v(\infty) = \gamma_w(\infty)\}.$$

It is clear that the strong stable manifold is included in the weak stable manifold and that the weak stable manifold is the same along the orbit of a vector by the geodesic flow. In fact, we will prove that each point in the weak stable manifold is on the strong stable manifold of one of these vectors, i. e. we have the equality

$$W^{so}(v) = \bigcup_{t\in\mathbb{R}} W^s(g_t(v)). \tag{4.7}$$

Let w any vector in $W^{so}(v)$ and denote $x = \pi(v)$, $y = \pi(w)$ and ξ the asymptotic class of the rays generated by v and w. Then y is in the horosphere centered at ξ and passing through $\gamma_v(t_0)$, where $t_0 = -b_{x,\xi}(y)$, so w is contained in $W^s(g_{t_0}(v))$. In effect,

$$b_{\gamma_v(t_0),\xi}(y) = b_{x,\xi}(y) + b_{\gamma_v(t_0),\xi}(x) = b_{x,\xi}(y) + t_0 = 0.$$

For every ξ in $M(\infty)$, we introduce a vector field v_ξ defined for all x in M by $v_\xi(x) = \dot{\sigma}_{x,\xi}(0)$. Then the weak stable manifold is described as

$$W^{so}(v) = \{v_\xi(z) \mid z \in M\}.$$

where $\xi = \gamma_v(\infty)$. The next proposition helps to understand stable manifolds.

Proposition 19. *Let M be a Hadamard manifold, x a point in M and ξ a point on the closure $M(\infty)$. Then the Busemann function $b_{x,\xi}$ at ξ based at x has regularity C^2 and we have*

$$\operatorname{grad} b_{x,\xi} = -v_\xi \tag{4.8}$$

and for all z in M and X in T_zM the covariant derivative of the previous fields is

$$D_X(\operatorname{grad} b_{x,\xi}) = -J'_X(0),$$

where J_X is the stable Jacobi field along the ray $\sigma_{z,\xi}$ with $J_X(0) = X$.

Sketch of the proof. For y in M consider the unitary radial field v_y centered at y, given by $v_y(z) = \dot{\sigma}_{z,y}(0)$ for all z in M. The first property is true if we replace horospheres for spheres, i. e. for all points x, y in M, we have

$$\operatorname{grad} b_{x,y} = -v_y.$$

Then applying the previous equality to a sequence of points converging to ξ and using the uniform convergence on compact sets to the Busemann function $b_{x,\xi}$ we get that it is C^1 and that $\operatorname{grad} b_{x,\xi} = -v_\xi$.

Let z, y be two points in M and X be a vector in T_zM, consider the Jacobi field J_X along $\sigma_{z,y}$ with conditions $J_X(0) = X$ and $J_X(d(z,y)) = 0$. Some computations show that

$$D_X v_y = J'_X(0)^\perp,$$

where $J'_X(0)^\perp$ indicates the component of $J'_X(0)$ orthogonal to $v_y(z)$. Again we have to apply the previous equality to a sequence of points converging to ξ and use properties of nonpositve curvature to obtain the covariant derivative of the gradient of the Busemann function on the left side. The right side converges to the stable Jacobi field along the ray $\sigma_{x,\xi}$ with initial condition X by Proposition 12. □

We deduce that the weak stable manifold at v is the set

$$W^{so}(v) = \{-\operatorname{grad} b_{x,\xi}(z) \mid z \in M\}$$

where $x = \pi(v)$ and $\xi = \gamma_v(\infty)$, and the stable manifold is

$$W^s(v) = \{-\operatorname{grad} b_{x,\xi}(z) \mid z \in M,\ b_{x,\xi}(z) = 0\}.$$

In other words, the stable manifold at v is the set of normal inward vectors to the horosphere centered at ξ passing through x. We also deduce that $W^s(v)$ is a C^1 submanifold of the unitary tangent bundle T^1M.

The set of tangent spaces to the stable manifold form the so-called tangent distribution E^s of W^s. These spaces are

$$E^s(v) = T_vW^s(v) = \{(X, Y) \in T_vT^1M \mid X \perp v,\ Y = J'_X(0)\},$$

because the first component has to be tangent to the horosphere, thus perpendicular to v by Proposition 19, and the second has to be $J'_X(0)$, where J_X is the unique stable Jacobi field along γ_v with initial condition X.

The stable manifold at v can be also defined as

$$W^s(v) = \{w \in T^1M \mid d_1(g_t(v), g_t(w)) \xrightarrow{t\to+\infty} 0\},$$

where d_1 is the Riemannian distance associated to the Sasaki metric on the unitary tangent bundle T^1M. If we add some hypothesis to the manifold M, the two definitions are equivalent because of the following lemma.

Lemma 6. *Let M be a Hadamard manifold with sectional curvature pinched between two negative constants. Then the distance d_1 associated to the Sasaki metric on the unitary tangent bundle T^1M is equivalent to the distance $\tilde{d}$ given by*

$$\tilde{d}(v, w) = d(\gamma_v(0), \gamma_w(0)) + d(\gamma_v(1), \gamma_w(1)) \quad \forall v, w \in T^1M.$$

This implies the equivalence of the two definitions because if two geodesic converge, $d(\gamma_v(t), \gamma_v(t)) \to 0$, then in the distance of the lemma we have

$$\tilde{d}(g_t(v), g_t(w)) = d(\gamma_{g_t(v)}(0), \gamma_{g_t(w)}(0)) + d(\gamma_{g_t(v)}(1), \gamma_{g_t(w)}(1)) =$$

$$d(\gamma_v(t), \gamma_w(t)) + d(\gamma_v(t+1), \gamma_w(t+1)) \to 0,$$

so they converge also in the distance d_1. The converse is clearly true.

The *unstable manifold at* v in T^1M is the set

$$W^u(v) = \{w \in T^1M \mid d(\gamma_v(t), \gamma_w(t)) \xrightarrow{t\to-\infty} 0\}.$$

All the results seen in this section are still valid for the unstable manifold if we reverse the time. In fact, since $g_{-t}(v) = -g_t(-v)$ for all real t and v in T^1M, we see that the unstable manifold at v is

$$W^u(v) = -W^s(-v).$$

Similarly, we can define the weak unstable manifold.

So far we have worked under the hypothesis that M is a Hadamard manifold. For a complete Riemannian manifold M with nonpositive curvature which is not simply connected, then its universal cover $\tilde{M}$ is a Hadamard manifold. We can define the strong (weak) stable (unstable) manifold at v in T^1M as the projection of the stable manifold at one lift $\tilde{v}$ of v. In the same way, horospheres, tangent distributions E^s, E^u, etc. are moved to the manifold M.

To finish the section, let us compute the horocycles (this is the name that receive horospheres in dimension 2) for the case of the hyperbolic plane. Some computation show that asymptotic classes of rays are identified with the endpoint of the geodesic contained on the border $\mathbb{R}\cup\{\infty\}$ of $\mathbb{H}$.

We compute the Busemann function at ∞ based at i. The geodesic joining i to infinity is $\sigma(t) = ie^t$. Let $z = x + iy$ in $\mathbb{H}$ and consider the point $\tau(t) = x + ie^t$. A computation using hyperbolic lengths shows that the distance between z and $\sigma(t)$ is bounded below by the distance between z and $\tau(t)$. On the other side, the length of the path joining z and $\tau(t)$ vertically and then joining $\tau(t)$ to $\sigma(t)$ is greater than the distance between z and $\sigma(t)$. Hence, we obtain

$$t - t_0 = d(z, \tau(t)) \leq d(z, \sigma(t)) \leq d(z, \tau(t)) + \frac{|x|}{e^t} = t - t_0 + \frac{|x|}{e^t},$$

where $t_0 = \log y$ and we assume that $t \geq t_0$. Therefore, the Busemann function at the point z is

$$b_{i,\infty}(z) = \lim_{t\to+\infty}(d(\sigma(t), z) - t) = -t_0 = -\log \mathrm{Im}(z).$$

We deduce that the horocycles centered at infinity are just horizontal lines. Using the fact that Busemann function are preserved by isometries, we can obtain the horocycles centered at a given point ξ in $\mathbb{R}$. They are the images of horizontal lines by homographies of $\mathrm{PSL}_2(\mathbb{R})$ that send ∞ to ξ, that is to say, circles tangent to the real line at the point ξ. They coincide with the projections of stable manifolds that we gave in Chapter 3.

4.2 The Hopf argument

Our ultimate goal is to prove the ergodicity of the geodesic flow with respect to the Liouville measure on a compact manifold of negative curvature. In other words, we have to show that every g_t-invariant function is constant almost everywhere. Morally, the proof has two steps: firstly, we show that an invariant function by the flow is invariant by the stable and unstable manifolds, in some sense that we will specify, and secondly, we will use a regularity property of the stable and unstable manifolds, called absolutely continuity, to conclude that if a function is invariant by these manifolds, then it is constant almost everywhere.

The key point of this argument is the absolutely continuity of the stable and unstable manifolds. The previous argument was first applied by E. Hopf in the case of surfaces of curvature -1 of finite volume [Hop36]. It was the first known proof of the ergodicity of geodesic flow in this situation, which is the reason why this way to proceed is known as the Hopf argument. We will add some remarks later to explain why the stable and unstable manifolds are absolutely continuous. The proof of absolutely continuity when the curvature is variable requires a more technical study of the stable and unstable manifolds, that we have already started and we will finish in the next sections.

4.2.1 Foliations

We will use the language of foliations, because it captures the essence of the problem. If $W = \{W_i\}_{i\in I}$ is a partition of the manifold X, for all x in X, we will denote by $W(x)$ the unique element of W that contains x. In addition, let B^k denote the closed unit ball of $\mathbb{R}^k$.

Definition 10. Let X^n be a manifold. A *k-dimensional C^0-foliation with C^1 leaves* is a partition W in k-dimensional connected C^1 submanifolds such that, for all x in X, there exists a neighborhood U of x and a homeomorphism $\varphi : B^k \times B^{n-k} \to U$ with $\varphi(0,0) = x$ satisfying, for all z in B^{n-k},

(i) the image by φ of the set $B^k \times \{z\}$ is the connected component $W_U(\varphi(0,z))$ of $W(\varphi(0,z)) \cap U$ containing $\varphi(0,z)$,

(ii) the map $\varphi(\,\cdot\,,z)$ is a C^1 diffeomorphism between B^k and $W_U(\varphi(0,z))$, depending continuously on z in the C^1-topology.

We will say that W is a C^1-foliation if the map φ is a diffeomorphism.

The set of stable and unstable manifolds are examples of C^0-foliations with C^1-leaves of the unitary tangent bundle T^1M, in Section 4.1.3 we discussed their regularity. We will refer to them as the stable foliation W^s, the unstable foliation W^u and analogously for the weak ones, W^{so} and W^{uo}.

Definition 11. Let X be a Riemannian manifold of finite volume and W be a partition. We say that a function $f : X \to \mathbb{R}$ is *invariant by the partition* W if there is a set Ω with complement of zero volume measure such that, for all x, y in Ω, the fact that $y \in W(x)$ implies that $f(y) = f(x)$.

Let Φ^t be a continuous flow on X. For all x in X, we define the stable set at x as the set

$$V^s(x) = \{y \in X \mid d(\Phi^t(x), \Phi^t(y)) \xrightarrow{t\to+\infty} 0\}.$$

The stable sets form a partition V^s. There are also the weak stable sets, defined as

$$V^{so}(x) = \bigcup_{t\in\mathbb{R}} \Phi^t(V^s(x)) = \bigcup_{t\in\mathbb{R}} W^s(\Phi^t(x)),$$

which form the weak stable partition V^{so}. Analogously we can define unstable partitions V^u and V^{uo}.

We recall the Birkhoff ergodic theorem.

Theorem 8. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, Φ^t be a measure preserving flow and $f : X \to \mathbb{R}$ and integrable function. Then, for μ-almost every x in X, the Birkhoff averages*

$$\frac{1}{T}\int_0^T f(\Phi^t(x))\,\mathrm{d}\mu$$

converge when T tends to infinity to a Φ^t-invariant function $\bar{f}$ in $L^1(X,\mu)$ and $\int \bar{f}\mathrm{d}\mu = \int f\mathrm{d}\mu$.

From this fact, we can deduce the first step of the Hopf argument. We will let $\mathrm{d}\mu$ be the volume form of the Riemannian manifold X and μ the volume measure.

Theorem 9. *Let X be a Riemannian manifold of finite volume and Φ^t a continuous flow on X preserving the volume measure. Let $f : X \to \mathbb{R}$ be a Φ^t-invariant function. Then f is invariant by the partitions V^s, V^u, V^{so} and V^{uo}.*

Proof. By modifying the function f on a set of zero measure if necessary, we can suppose that it is strictly invariant, i. e. $f \circ \Phi^t = f$ for all real t.

By Lusin's theorem, given a number $\varepsilon > 0$, we can fix a Borel set F such that $\mu(X \setminus F) < \varepsilon$ where the function f is uniformly continuous.

The Birkhoff ergodic theorem applied to the indicator function of the set F says that the average time spent by the orbit of x in the set F

$$\tau_F(x) = \lim_{T\to+\infty} \frac{1}{T}\lambda(\{t \in [0,T] \mid \Phi^t(x) \in F\})$$

is defined for a.e. x in X, λ being the Lebesgue measure. In addition, the set $A_F = \{\tau_F > \frac{1}{2}\}$ is invariant, because τ_F is.

Suppose that x and y are two points in A_F such that $y \in V^{so}(x)$. There exists a point z in $V^s(x)$ and a real number t_0 such that $y = \Phi^{t_0}(z)$. Then z is also in A_F because τ_F is invariant by the flow. Since $\tau_F(x), \tau_F(z) > \frac{1}{2}$, there exist a sequence of real numbers t_n converging to infinity such that $\Phi^{t_n}(x), \Phi^{t_n}(z) \in F$ for all n. We also know that the distance between the orbits of x and z goes to zero because they are in the same stable set. Because of the uniform continuity on F and the invariance of f, we have

$$|f(x) - f(y)| = |f(x) - f(z)| = \left|f(\Phi^{t_n}(x)) - f(\Phi^{t_n}(z))\right| \xrightarrow{n\to+\infty} 0.$$

The property holds in the set A_F. Observe that its complement is $A_F^c = \{\tau_{F^c} \geq \frac{1}{2}\}$ and it has measure

$$\mu(A_F^c) = \int_{A_F^c} 1 \,\mathrm{d}\mu \leq \int_{A_F^c} 2\tau_{F^c} \,\mathrm{d}\mu \leq 2\int_X \tau_{F^c} \,\mathrm{d}\mu = 2\mu(F^c) < 2\varepsilon.$$

Therefore we can consider a sequence of sets A_n such that $\mu(A_n^c) < 2^{-n}$, for all natural n, where the property holds. Then the sequence of sets $B_n = \cap_{k\geq n} A_k$ is increasing and the measure of their complements goes to zero. That allows to conclude that there is a set of full measure where the property is true.

The proof for the weak stable partition is analogous. The case of strong partitions is clear from the weak ones. □

In the case of the geodesic flow, stable and unstable partitions correspond to the stable and unstable foliations W^s and W^u and the involved measure is the Liouville measure.

4.2.2 Absolutely continuity

The second step in Hopf argument is the passage from the invariance on stable and unstable foliations to the fact that the function is constant at least on a neighborhood using a version of the Fubini's theorem. But the foliations have not enough regularity a priori, so we will need to introduce a new property, the absolute continuity, that allows to complete the argument.

A *transversal* L of a k-dimensional foliation W on some manifold X is a $(n-k)$-submanifold of X such that at every point x in L the tangent spaces of $W(x)$ and L are a direct sum $T_xL \oplus T_xW(x) = T_xX$. The induced volume form on a submanifold Y of X will be denoted by $\mathrm{d}\mu_Y$.

Definition 12. Let X be a Riemannian manifold, W be a k-dimensional C^0-foliation with C^1-leaves We say that W is absolutely continuous if, for every transversal L and for every open set U of X such that

$$U = \coprod_{x\in L\cap U} W_U(x),$$

where $W_u(x)$ is the connected component of $W(x) \cap U$ containing x and it is diffeomorphic to B^k, there exists a family of positive measurable functions $\delta_x : W_U(x) \to \mathbb{R}$, $x \in L \cap U$, such that for every measurable subset A of U we have

$$\mu(A) = \int_{L \cap U} \int_{W_U(x)} \mathbf{1}_A(y)\delta_x(y) \, \mathrm{d}\mu_{W_U(x)}(y) \, \mathrm{d}\mu_{L\cap U}(x).$$

The definition is a way of saying that the volume measure on the manifold can be locally decomposed as a sum of measures on each leaf of the foliation that are equivalent to the induced volume measure. We have defined this new property because of the following essential fact.

Proposition 20. *Let X be a connected Riemannian manifold and let W_1, W_2 be two absolutely continuous foliations of X such that at each point x in X we have $T_xW_1(x) \oplus T_xW_2(x) = T_xX$. Let $f : M \to \mathbb{R}$ be a measurable function. If the function f is invariant by the foliations W_1 and W_2, then it is constant almost everywhere.*

Proof. Let N_1, N_2 be null subsets such that for all $x, y \in N_1, N_2$, the fact that $y \in W_1(x), W_2(x)$ implies $f(x) = f(y)$. We set $N = N_1 \cup N_2$ and consider the full measure subset $\bar{X} = X \setminus N$ where f is constant on the leaves of both W_1 and W_2.

Let x be a point in X. By hypothesis $W_1(x)$ is a transversal of the foliation W_2. There exists a neighborhood U of x satisfying

$$U = \coprod_{y \in W_{1U}(x)} W_{2U}(y).$$

The volume measure of $N \cap U$ is zero. By the absolute continuity of W_2 we deduce that for a.e. y in $W_{1U}(x)$ the set $W_{2U}(y) \setminus N$ has full $\mu_{W_{2U}(y)}$-measure. We fix an y that has an open neighborhood $V \subset U$ of y containing x and such that V is the union of the local leaves $W_{1V}(z)$ for z in $W_{2V}(y)$. On the set $W_{2U}(y) \setminus N$ the function f is constant equal to c in $\mathbb{R}$.

We consider the subset

$$\Omega = \bigcup_{z \in W_{2V}(y) \setminus N} W_{1V}(z)$$

of V. Then, since $W_2(y)$ is a transversal of the foliation W_1, by the absolute continuity we get that Ω has full measure in V. Every point w in $\Omega \setminus N$ is in the W_1-leaf of some z in $W_{2V}(y) \setminus N$. Since the function is constant in the leaves of W_1 outside N, it follows $f(w) = f(z) = c$. Finally, the set $\Omega \setminus N$ has full measure in the neighborhood V of x, hence f is locally a. e. constant. We conclude that f is a.e constant on X by connectedness. $\square$

We want to apply the previous proposition to the case of the geodesic flow. The foliations we use have to be *transversal*, in the sense that at each point the tangent spaces of the two leaves are in direct sum, and have complementary dimensions. The tangent space at v of the unitary tangent bundle is decomposed in three subspaces, namely, the stable space $E^s(v)$, the unstable space $E^u(v)$ and the space of the direction of the flow, as we have seen in Section 4.1.3. The tangent space of the weak stable manifolds is the sum of the corresponding strong space and the direction of the flow. Therefore, Proposition 20 has to be applied to foliations W^s and W^{uo}, or vice versa. We already know that any g_t-invariant function is invariant by the two foliations and we are going to prove that foliations W^s and W^u are absolutely continuous. The following result will allow to deduce that W^{uo} is also absolutely continuous.

Let W, W_1, W_2 be three foliations of X of dimensions d, d_1, d_2 with $d = d_1 + d_2$. We say that W_1 and W_2 are *integrable* and that W is the *integral hull* of W_1 and W_2 if they are transversal and, for all x in X, we have

$$W(x) = \cup_{y \in W_1(x)} W_2(y) = \cup_{y \in W_2(x)} W_1(y).$$

Lemma 7. *Let W_1 and W_2 two integrable C^0-foliations with C^1-leaves of a Riemannian manifold X with integral hull W. If W_1 is C^1 and W_2 is absolutely continuous, then W is absolutely continuous.*

Proof. Let L be a transversal of W and U an open set that is the union of the local leaves $W_U(x)$ where x is in $L \cap U$. Then the set $\tilde{L} = \cup_{x \in L \cap U} W_{1U}(x)$ is a transversal for the foliation W_2. By hypothesis, W_2 is absolutely continuous, so for all measurable subset A of U we have

$$\mu(A) = \int_{\tilde{L}} \int_{W_{2U}(y)} \mathbf{1}_A(z) \delta_y(z) \, \mathrm{d}\mu_{W_{2U}(y)}(z) \, \mathrm{d}\mu_{\tilde{L}}(y).$$

The restriction of W_1 to $\tilde{L}$ is a C^1-foliation. Then there exists a continuous Jacobian j such that for all integrable function h we have

$$\int_{\tilde{L}} h(y) \, \mathrm{d}\mu_{\tilde{L}}(y) = \int_{L \cap U} \int_{W_{1U}(x)} h(y) j(x, y) \, \mathrm{d}\mu_{W_{1U}(x)}(y) \, \mathrm{d}\mu_{L \cap U}(x).$$

The restriction of W_1 at each leaf $W(x)$ is a C^1-foliation. So there exist a family of Jacobians η_x such that for all integrable function g, we have the equality

$$\int_{W_U(x)} g(z) \mathrm{d}\mu_{W_U(x)}(z) = \int_{W_{2U}(x)} \int_{W_{1U}(y)} g(z) \eta_x(y, z) \mathrm{d}\mu_{W_{1U}(y)}(z) \mathrm{d}\mu_{W_{2U}(x)}(y).$$

Changing the order of integrals by Fubini and combining the three formulas we can express the measure of A as an integral on L of an integral on the leaves of W and we can conclude that W is absolutely continuous. □

By Equation 4.7, the weak unstable foliation W^{uo} is the integral hull of W^u and the foliation of the unitary tangent bundle by orbits of the geodesic flow, which is differentiable. If we show that the strong unstable foliation is absolutely continuous, we will be able to deduce that the weak one is absolutely continuous too.

4.2.3 The case of surfaces with curvature -1

Let us come back to the manifolds of constant curvature. In Chapter 3 we studied the stable and unstable manifolds of the hyperbolic plane and we saw that there is a coordinate system (s, t, u) of the unitary tangent manifold. This system has the vertical upward vector v_0 at i as origin. If we fix a value of u then s, t parametrize the weak stable manifold of the point $h_u^-(v_0)$. On the contrary, if we fix values of s and t, the parameter u moves along the strong unstable manifold at $g_t(h_s^+(v_0))$.

We showed that in these coordinates the Liouville measure is expressed as an integral of a Jacobian, that we computed, on the weak stable and strong stable manifolds, because s, t, u are all arc length parameters. Since we already know that a g_t-invariant function is invariant by stable and unstable manifolds, the argument used in the proof of Proposition 20 allows to conclude that the function is constant almost everywhere. As we have said, the proof of ergodicity in constant curvature is simpler and it came before than the general case.

4.2.4 Transversal absolute continuity

All that remains to do is to prove the ergodicity of geodesic flow is to show the absolute continuity of stable and unstable foliations. In fact, we will prove that they satisfy a stronger property than absolute continuity that we will define next.

Definition 13. Let W be a foliation of a Riemannian manifold X, x_1 be a point X, x_2 a point in the leaf $W(x_1)$ and L_1, L_2 two transversals containing the points x_1, x_2, respectively. There are small enough neighborhoods U_1, U_2 of x_1, x_2 in L_1, L_2, respectively, such that for every point y in U_1 the intersection of the leaf $W(y)$ and the set U_2 is a unique point. The *Poincaré map* is the homeomorphism $p : U_1 \to U_2$ that sends y to this unique point.

We say that the foliation W is transversally absolutely continuous if for every Poincaré map $p : U_1 \to U_2$ there is a positive measurable function $q : U_1 \to \mathbb{R}$, called the *Jacobian* of p, such that for every measurable subset A of U_1 we have

$$\mu_{L_2}(p(A)) = \int_{U_1} \mathbf{1}_A(y) q(y) \, \mathrm{d}\mu_{L_1}(y).$$

Next we will establish the relation between the two properties.

Proposition 21. *Let W be a foliation of a Riemannian manifold X. If W is transversally absolutely continuous, then it is absolutely continuous.*

Proof. Let L be a transversal, U be an open set of X that is the union of the local leaves $W_U(y)$ where y is in $L \cap U$ and x be a point in $L \cap U$. The transversal L can be extended to a C^1-foliation F such that $F(x) = L$ and

$$U = \coprod_{y \in W_U(x)} F_U(y).$$

This foliation is absolutely continuous and transversally absolutely continuous because it is differentiable. We apply the absolutely continuity for the transversal $W_U(x)$ of F. There exists a family of positive measurable functions $\bar{\delta}_y : F_U(y) \to \mathbb{R}$ such that for every measurable subset A of U, we have

$$\mu(A) = \int_{W_U(x)} \int_{F_U(y)} \mathbf{1}_A(z) \bar{\delta}_y(z) \; \mathrm{d}\mu_{F_U(y)}(z) \, \mathrm{d}\mu_{W_U(x)}(y).$$

Now we apply the transversally absolutely continuity of W between the transversals $F_U(x) = L \cap U$ and $F_U(y)$. We denote by p_y the Poincaré map between the two tranversals and q_y its Jacobian. We can write the equality

$$\int_{F_U(y)} \mathbf{1}_A(z) \bar{\delta}_y(z) \; \mathrm{d}\mu_{F_U(y)}(z) = \int_{L \cap U} \mathbf{1}_A(p_y(s)) \bar{\delta}_y(p_y(s)) q_y(s) \; \mathrm{d}\mu_{L \cap U}(s).$$

We put the previous expression in the first equation and change the order of integration by Fubini,

$$\mu(A) = \int_{L \cap U} \int_{W_U(x)} \mathbf{1}_A(p_y(s)) \bar{\delta}_y(p_y(s)) q_y(s) \; \mathrm{d}\mu_{W_U(x)}(y) \, \mathrm{d}\mu_{L \cap U}(s).$$

Finally, we use the transversally absolutely continuity of F between the transversals $W_U(x)$ and $W_U(s)$. We let $\bar{p}_s$ denote the Poincaré map and $\bar{q}_s$ its Jacobian. We notice that if $y = \bar{p}_s^{-1}(r)$, where $r \in U$, then $p_y(s) = r$. Hence, it follows that

$$\mu(A) = \int_{L \cap U} \int_{W_U(s)} \mathbf{1}_A(r) \bar{\delta}_{\bar{p}_s^{-1}(r)}(r) q_{\bar{p}_s^{-1}(r)}(s) \bar{q}_s^{-1}(r) \; \mathrm{d}\mu_{W_U(s)}(r) \, \mathrm{d}\mu_{L \cap U}(s).$$

Therefore, the foliation W is absolutely continuous. $\square$

4.3 Anosov flows

In this section, we study a class of flows introduced by D. V. Anosov that are a generalization of the geodesic flow, copying the hyperbolicity of its differential. Let M be a compact Riemannian manifold and Φ^t a differentiable flow on M. If the flow is generated by a vector field X, i. e. $X(x) = \frac{d}{dt}|_{t=0}\Phi^t(x)$, we will denote the direction of the flow by $E^o(x) := \langle X(x) \rangle$.

Definition 14. We say that Φ^t is Anosov if it has no fixed points and for every x in M there exist two proper Φ^t-invariant subspaces $E^s(x)$, $E^u(x)$ of T_xM, and constants $C > 0$, $\lambda \in (0,1)$ such that

(i) $E^s(x) \oplus E^u(x) \oplus E^o(x) = T_xM$,

(ii) $\|d_x\Phi^t(u)\| \le C\lambda^t\|u\|$, $\forall t \ge 0$, $\forall u \in E^s(x)$,

(iii) $\|d_x\Phi^{-t}(u)\| \le C\lambda^t\|u\|$, $\forall t \ge 0$, $\forall u \in E^u(x)$.

The condition of not having fixed points is equivalent to say that $E^o(x)$ has always dimension 1, or equivalently, $X(x) \neq 0$ for all x in M. The Φ^t-invariance of $E^{s,u}(x)$ means that $d_x\Phi^t(E^{s,u}(x)) = E^{s,u}(\Phi^t(x))$. For an Anosov flow, spaces $E^s(x)$ and $E^u(x)$ are uniquely determined by conditions 2 and 3.

We are interested in a distance between subspaces of tangent spaces. We will give the definition of one distance convenient for our purpose, but in the literature we can find other distances equivalent to this one. Let x, y be two points in M, F a subspace of T_xM and H a subspace of T_yM. Let $T_{x,y} : T_xM \to T_yM$ be the parallel transport on the minimizing geodesic segment $[x, y]$ between x and y. Define the distance

$$D(F, H) = d(x, y) + d_{Gr}(T_{x,y}F, H)$$

where $d(x, y)$ stands for the Riemannian distance and d_{Gr} is a distance on the set of subspaces of the vectors space T_yM. We can define it in the following manner: let $(V, \|\cdot\|)$ a normed vector space, U, W two subspaces and P_U, P_W the orthogonal projections on each space, we set

$$d_{Gr}(U, W) = |||P_U - P_W|||$$

where $|||\cdot|||$ is the operator norm associated to $\|\cdot\|$. Having defined the distance, we can announce the following property.

Lemma 8. *$E^s(x)$ and $E^u(x)$ depend continuously on x.*

Proof. Let $x_n \to x$ be a converging sequence. We will start by showing that there is a subsequence such that $E^s(x_{n_k}) \to E^s(x)$ and $E^u(x_{n_k}) \to E^u(x)$. First, extracting a subsequence if necessary, we suppose that $\dim E^s(x_n) = k$ and $\dim E^u(x_n) = l$ for all n, where $k + l + 1 = \dim M$. Consider the family of parallel transported subspaces $\{T_{x_n,x}E^s(x_n)\}_n$ living in the Grassmannian $Gr_k(T_xM)$ of k-dimensional subspaces of T_xM. By the compacity of the Grassmannian, there is a converging subsequence

$$T_{x_{n_k},x}E^s(x_{n_k}) \to H^s.$$

In addition, as $x_{n_k} \to x$, we see from the definition of the distance between subspaces with different basepoints that the sequence $E^s(x_{n_k})$ itself converges to the subespace H^s of T_xM. Repeating the argument for the other subspace we find a sequence such that $\lim E^s(x_{n_k}) = H^s$ and $\lim E^u(x_{n_k}) = H^u$. A vector v in H^s is a limit of vectors satisfying condition 2 in Definition 14. Since $d_z\Phi^t(w)$ is continuous on z in M and w in T_zM, v will satisfy the condition too. Therefore, we have $H^s \subset E^s(x)$ and $H^u \subset E^u(x)$. Finally, by dimension, we deduce the equalities $\lim E^s(x_{n_k}) = H^s = E^s(x)$ and $\lim E^u(x_{n_k}) = H^u = E^u(x)$.

This is enough to prove the continuity. In effect, suppose that the sequence $E^s(x_n)$ does not converge to $E^s(x)$. For some $\delta > 0$, we can find a subsequence n_j such that $D(E^s(x_{n_j}), E^s(x)) \geq \delta$. By compacity, the last sequence has a converging subsequence $E^s(x_{n_{j_i}})$ to some space H. Then, the inequality $D(H, E^s(x)) \geq \delta$ holds. Finally, applying the first part of the proof to the last sequence, there is a subsequence of $E^s(x_{n_{j_i}})$ converging to $E^s(x)$, but this contradicts the lower bound $D(H, E^s(x)) \geq \delta$. □

We can conclude that E^s and E^u are continuous Φ^t-invariant distributions, called the *stable distribution* and the *unstable distribution*, respectively. It turns out that these distributions satisfy a stronger regularity condition that we will see next.

First, we will slightly modify the metric of the manifold to simplify the computations. Fix numbers $\beta \in (\lambda, 1)$ and $T > 0$. Every vector v in T_xM has a decomposition $v = v_s + v_u + v_o \in E^s(x) \oplus E^u(x) \oplus E^o(x)$. Define

$$|v_s| = \int_0^T \frac{\|d_x\Phi^\tau(v_s)\|}{\beta^\tau} d\tau, \quad |v_u| = \int_0^T \frac{\|d_x\Phi^{-\tau}(v_u)\|}{\beta^\tau} d\tau,$$

$$|v_o| = \sup_{t\in\mathbb{R}} \left\|d_x\Phi^t(v_o)\right\|, \quad |v|^2 = |v_s|^2 + |v_u|^2 + |v_o|^2.$$

By compacity there are constants $C_1, C_2 > 0$ such that for all z in M, we have $C_1 \leq \|X(z)\| \leq C_2$. Hence, for all x in M, for all t in $\mathbb{R}$,

$$\frac{C_1}{C_2}\|v_o\| \leq \left\|d_x\Phi^t(v_o)\right\| \leq \frac{C_2}{C_1}\|v_o\|.$$

It is not difficult to verify that $|\cdot|$ is a norm on each tangent space T_xM, and it is equivalent to $\|\cdot\|$ because all norms on a finite vector spaces are equivalent. The new norm induces a Riemannian metric on M, which is equivalent to the original one because of the compacity. From the definition, it is obvious that subspaces $E^s(x), E^u(x), E^o(x)$ are pairwise orthogonal by the new metric.

We observe that

$$\left|d_x\Phi^t(v_s)\right| = \int_0^T \frac{\left\|d_x\Phi^{\tau+t}(v_s)\right\|}{\beta^\tau} d\tau = \beta^t \int_t^{t+T} \frac{\|d_x\Phi^\tau(v_s)\|}{\beta^\tau} d\tau =$$

$$= \beta^t \left(|v_s| + \int_T^{t+T} \frac{\|d_x\Phi^\tau(v_s)\|}{\beta^\tau} d\tau - \int_0^t \frac{\|d_x\Phi^\tau(v_s)\|}{\beta^\tau} d\tau\right).$$

But

$$\int_T^{t+T} \frac{\|d_x\Phi^\tau(v_s)\|}{\beta^\tau} d\tau = \int_0^t \frac{\left\|d_x\Phi^{\tau+T}(v_s)\right\|}{\beta^{\tau+T}} d\tau \leq \frac{C\lambda^T}{\beta^T}\int_0^t \frac{\|d_x\Phi^\tau(v_s)\|}{\beta^\tau} d\tau,$$

so, if we suppose that T is big enough so that $C(\lambda/\beta)^T < 1$, then we get $\left|d_x\Phi^t(v_s)\right| \leq \beta^t |v_s|$. Similarly, we have $\left|d_x\Phi^{-t}(v_u)\right| \leq \beta^t |v_u|$ for all nonnegative t, and it is clear that $\left|d_x\Phi^{-t}(v_o)\right| = |v_o|$ for t in $\mathbb{R}$. With this new metric, called *adjusted metric*, the flow is still Anosov, with normalized constant and orthogonality between stable, unstable and flow directions.

From now on all Anosov flows will be considered with an adjusted metric and constants $C = 1$ and $\lambda \in (0, 1)$. Next, we look at the evolution of subspaces under the flow.

Lemma 9. *Let Φ^t be an Anosov flow on M and x a point of M. Then, for every $0 < \theta < \pi/2$, there exists a constant $K > 0$ such that for every subspace H of T_xM of the same dimension as $E^s(x)$ that satisfies*

$$\min_{v \in H\setminus\{0\}} |\angle(v, E^u(x) \oplus E^o(x))| \geq \theta$$

then, for $t \geq 0$,

$$D(d_x\Phi^{-t}(H), E^s(\Phi^{-t}(x))) \leq K\lambda^t D(H, E^s(x)). \tag{4.9}$$

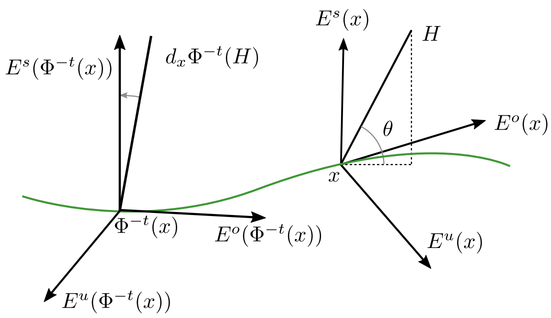

Figure 4.1: Evolution of subspaces under the flow.

Proof. The proof relies on the fact that the stable component of H decreases with time t and the rest remains bounded, thus the angle between the subspaces H and E^s reduces when time goes back (see Figure 4.1). Let v in H with norm $\|v\| = 1$. It decomposes as $v = v_s + v_u + v_o \in E^s(x) \oplus E^u(x) \oplus E^o(x)$. From the Anosov flow conditions we have, for every nonnegative t,

$$\|d_x\Phi^{-t}(v_s)\| \geq \lambda^{-t}\|v_s\|,$$

$$\|d_x\Phi^{-t}(v_u)\| \leq \lambda^t\|v_u\| \leq \|v_u\|.$$

Denoting $v_{uo} = v_u + v_o$, it follows the inequality $\left\|d_x\Phi^{-t}(v_{uo})\right\| \leq \|v_{uo}\|$. The tangent of the angle formed by the vector $d_x\Phi^{-t}(v)$ with the stable subspace is

$$\frac{\left\|d_x\Phi^{-t}(v_{uo})\right\|}{\|d_x\Phi^{-t}(v_s)\|} \leq \lambda^t \frac{\|v_{uo}\|}{\|v_s\|}$$

Now, the quantity $\|v_{uo}\|$ is the cosine of v with the stable subspace, so it is bounded by a constant multiple of the distance $D(H, E^s(x))$. By hypothesis, $\|v_s\|$ is bounded below by a constant ($\sin\theta$). Hence, we obtain

$$\frac{\left\|d_x\Phi^{-t}(v_{uo})\right\|}{\|d_x\Phi^{-t}(v_s)\|} \leq A\lambda^t D(H, E^s(x))$$

for some constant $A > 0$. Since the inequality is valid for all v in H of norm $\|v\| = 1$ and the supremum of all the tangents of angles between vectors of H and the subspace $d_x\Phi^t E^s(x)$ bounds the distance between subspaces (up to some constant), we obtain the desired inequality. $\square$

We say that a distribution $\{E(x)\}_{x\in M}$, where each $E(x)$ is a subspace of the tangent space T_xM, is *Hölder continuous* if there exist constants $A, \alpha > 0$ such that, for all points x, y in M,

$$D(E(x), E(y)) \leq A\, d(x, y)^\alpha.$$

Theorem 10. *Let Φ^t be a C^2 Anosov flow. Then the distributions E^s, E^u, $E^{so} := E^s \oplus E^o$, $E^{uo} := E^u \oplus E^o$ are Hölder continuous.*

Proof. Let us first prove the property for the stable distribution E^s. Recall that the distance between two stable subspaces is $D(E^s(x), E^s(y)) = d_{Gr}(E^s(x), T_{y,x}E^s(y)) + d(x,y)$, so we only have to deal with the first term, since the second is clearly Hölder continuous. We will denote $\Phi = \Phi^1$.

Let x be a point in M. For a natural number m, we consider coordinate balls V_k centered at $\Phi^k(x)$, for $0 \le k \le m$. Let $\varepsilon_1 > 0$ be small enough so that $\Phi^k(B(x,\varepsilon_1)) \subset V_k$, $k = 0, \dots, m$, where $B(x,\varepsilon_1)$ is the ball of radius ε_1 centered at x. For each y in $B(x,\varepsilon_1)$, let A_k denote the matrix of $d_{\Phi^{m-k}(x)}\Phi^{-1}$ in the chart of V_{m-k} and B_k denote the matrix of $T_{y,x} \circ d_{\Phi^{m-k}(y)}\Phi^{-1} \circ T_{x,y}$ in the same chart. In addition, if v is in $E^s(\Phi^m(y))$, for $0 \le k \le m$, we set

$$v_k = T_{\Phi^{m-k}(y),\Phi^{m-k}(x)} \circ d_{\Phi^m(y)}\Phi^{-k}(v),$$

$$v_k = v_k^s + v_k^{uo} \in E^s(\Phi^{m-k}(x)) \oplus E^{uo}(\Phi^{m-k}(x)).$$

In Figure 4.2 we can see the situation and these notations.

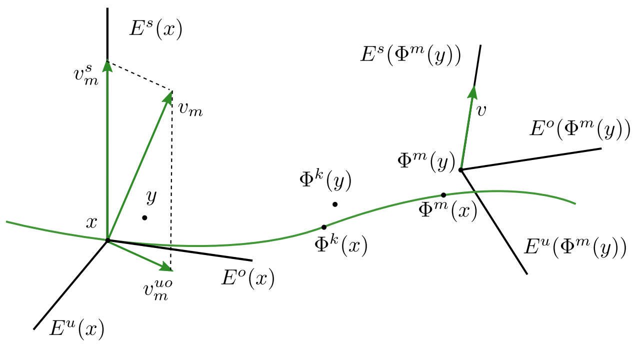

Figure 4.2: Notations for the proof of Hölder continuity.

Consider constants $\kappa \in (\sqrt{\lambda}, 1)$,

$$D > \max\left\{\sup_{z\in M} \|d_z\Phi\|, \frac{1}{\kappa}\right\}, \quad M \ge \sup_{v\in TM} \left\|d_v^2\Phi^{-1}\right\|,$$

that exist because Φ^t is C^2 and M is compact. Fix also $\delta < (1-\sqrt{\lambda})/2$.

The distribution E^s is continuous and M is compact, so E^s is uniformly continuous. Consequently, there exists a constant $\varepsilon_0 > 0$ such that if z, z' are two points in M at distance $d(z,z') < \varepsilon_0$ then we have

$$\sup_{v\in E^s(z')} \frac{\left\|\left(T_{z',z}v\right)^{uo}\right\|}{\left\|\left(T_{z',z}v\right)^{s}\right\|} \le \delta,$$

because the left side is equivalent to the distance between $E^s(z)$ and the parallel transport of $E^s(z')$ when they are small.

Now, take

$$\varepsilon < \min\left\{\frac{1-\sqrt{\lambda}}{2\lambda M}, \frac{\delta(\kappa-\sqrt{\lambda})}{\sqrt{\lambda}M}, \varepsilon_0, \varepsilon_1\right\}.$$

We claim that for all y in M at distance $d(y,x) < \varepsilon$ and for all integer $m \geq 0$ such that $D^m d(x,y) < \varepsilon$, setting the discussed notations, we have

$$\frac{\|v_k^{uo}\|}{\|v_k^s\|} \leq \delta\kappa^k \text{ for } 0 \leq k \leq m,$$

for all $v \in E^s(\Phi^m(y))$. The proof is done by induction. The case $k = 0$ is true by the choice of ε. Suppose it is true for $k \leq m-1$ and we show that it also holds for $k+1$.

The Anosov conditions imply $\|A_k v_k^s\| \geq \lambda^{-1} \|v_k^s\|$ and $\|A_k v_k^{uo}\| \leq \|v_k^{uo}\|$. In addition, we can write

$$v_{k+1} = B_k v_k = A_k v_k + (B_k - A_k)v_k = A_k v_k^s + A_k v_k^{uo} + (B_k - A_k)v_k,$$

therefore

$$\frac{\|v_{k+1}^{uo}\|}{\|v_{k+1}^s\|} \leq \frac{\|A_k v_k^{uo}\| + \|B_k - A_k\| \|v_k\|}{\|A_k v_k^s\| - \|B_k - A_k\| \|v_k\|} \leq \frac{\delta\kappa^k \|v_k^s\| + M\varepsilon D^{-k} \|v_k\|}{\lambda^{-1} \|v_k^s\| - M\varepsilon \|v_k\|}$$

where we used that

$$\|B_k - A_k\| \leq M\, d(\Phi^{m-k}(x), \Phi^{m-k}(y)),$$
$$d(\Phi^{m-k}(x), \Phi^{m-k}(y)) \leq D^{m-k}\, d(x,y) < \varepsilon D^{-k} < \varepsilon.$$

Next, since $\|v_k\| \leq \|v_k^s\| + \|v_k^{uo}\| \leq \|v_k^s\| + \delta\kappa^k \|v_k^s\| \leq \|v_k^s\| + \delta\, \|v_k\|$, we obtain

$$\frac{\|v_{k+1}^{uo}\|}{\|v_{k+1}^s\|} \leq \frac{\delta\kappa^k \|v_k\| + M\varepsilon\kappa^k \|v_k\|}{\lambda^{-1}(1-\delta) \|v_k\| - M\varepsilon \|v_k\|} = (\delta + M\varepsilon)\sqrt{\lambda}\,\kappa^k \frac{\sqrt{\lambda}}{1 - \delta - \lambda M\varepsilon}.$$

The fraction is less than 1 because of the choice of δ and ε. The choice of ε ensures that $(\delta + M\varepsilon)\sqrt{\lambda} < \delta\kappa$ as well, and we get the desired inequality.

In particular, if m is the integer part of $(\log\varepsilon - \log d(x,y))/\log D$,

$$\frac{\|v_m^{uo}\|}{\|v_m^s\|} \leq \delta\kappa^m \leq \delta\kappa^{\frac{\log\varepsilon - \log d(x,y)}{\log D} - 1} = \delta\kappa^{\frac{\log\varepsilon}{\log D} - 1}\, d(x,y)^{-\frac{\log\kappa}{\log D}}.$$

This is valid for every v in $E^s(\Phi^m(y))$, which covers all the vectors v_m in $T_{y,x}E^s(y)$, thus $d_{Gr}(E^s(x), T_{y,x}E^s(y)) \leq A\, d(x,y)^\alpha$ for some $A > 0$ and $\alpha = -\log\kappa/\log D > 0$. Modifying the constants, this is valid not only if $d(x,y) < \varepsilon$, but for all y in M, because M is compact. Again using the compacity, we find constants that do not depend on x in M. This proves the Hölder continuity of E^s.

Reversing the time, we obtain the property for the distribution E^u. The distribution E^o has regularity C^1, and in particular Hölder. Finally, E^{so} and E^{uo} are Hölder because they are sums of Hölder distributions. □

To finish the section we recall the results in Section 4.1 to show that the geodesic flow g^t on the unitary tangent bundle of a compact manifold with negative curvature satisfies the Anosov conditions. The stable and unstable subspaces in the definition of Anosov flow coincide with the ones we had described geometrically as

$$E^s(v) = \{(X,Y) \in T_v T^1 M \mid X \perp v,\, Y = {J_X^s}'(0)\},$$

$$E^u(v) = \{(X,Y) \in T_v T^1 M \mid X \perp v,\, Y = -{J_X^u}'(0)\},$$

where J_X^s and J_X^u are the stable and unstable Jacobi fields along the geodesic γ_v with initial condition X. The first of the Anosov conditions is clearly satisfied because spaces $E^s(v)$, $E^u(v)$ and $E^0(v) = \langle (v,0) \rangle$ are disjoint and their dimensions are complementary.

Let us show that the second condition is satisfied. Given v in T^1M and (X, Y) in $E^s(v)$, the stable Jacobi field J_X satisfies $J_X(0) = X$ and $J'_X(0) = Y$, so we have for $t \geq 0$

$$\|d_v g_t(X,Y)\| = \left\|(J_X(t), J'_X(t))\right\| = \sqrt{\|J_X(t)\|^2 + \left\|J'_X(t)\right\|^2} \leq \|(X,Y)\| \sqrt{1+b}\, e^{-at}$$

by Proposition 13, where $-a^2$ and $-b^2$ are the upper and lower bounds of the sectional curvature. The third Anosov condition follows from the analogous result for unstable Jacobi fields.

4.4 Proof of the ergodicity of the geodesic flow

In this section we will complete the proof of the ergodicity of the geodesic flow for a compact manifold. We will need the following result from measure theory.

Lemma 10. *Let X, Y be two compact metric spaces and $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be two Borel measured spaces. Let $p : X \to Y$ and $p_n : X \to Y$, for all natural n, be continuous maps. Suppose that*

(i) the maps p_n, $n \in \mathbb{N}$, and p are homeomorphisms onto their image,

(ii) the sequence p_n converges uniformly to p,

(iii) there is a constant C such that for all measurable subset A of X, we have $\nu(p_n(A)) \leq C\mu(A)$.

Then, for all measurable subset A of X we have $\nu(p(A)) \leq C\mu(A)$.

Finally, we are prepared to show the transversally absolutely continuity of the stable and unstable foliations.

Theorem 11. *Let M be a C^3 compact Riemannian manifold with negative sectional curvature. Then the stable and unstable foliations, W^s and W^u, of the unitary tangent bundle T^1M are transversally absolutely continuous.*

Proof. We will treat the case of the stable foliation, the other case is analogous.

We need to show that for every pair of points v_1, v_2 in T^1M such that $v_2 \in W^s(v_1)$, for every pair of transversals L_1, L_2 to W^s containing the points v_1, v_2, respectively, the Poincaré map $p : U_1 \to U_2$, defined between two neighborhoods of the points v_i in L_i, has a Jacobian $q : U_1 \to \mathbb{R}$. Figure 4.3 shows how the Poincaré map acts on the disk.

The set of inward normal vectors of a sphere in M of radius k is a submanifold of the unitary tangent bundle. The set of all these submanifolds with k fixed forms a foliation, that will be denoted by Σ_k. In Section 4.1.3 we have seen that spheres of increasing radius normal to a vector converge uniformly on compact sets to the horosphere associated to the vector. Thus, leaves of Σ_k converge uniformly on compact sets to leaves of the stable foliations W^s when k tends to infinity. We will let $k \in \mathbb{N}$ tend to infinity.

If k is big enough, L_1 and L_2 will be transversals of the foliation Σ_k at least on the neighborhoods U_1 and U_2. We consider the Poincaré map $p_k : U_1 \to U_2$ of the foliation Σ_k. We want to apply Lemma 10 to the maps p_k and p on properly chosen compacts with the induced volume measure. The first condition it is clearly satisfied and the second is a consequence of the uniform convergence on compact sets of the leaves of Σ_k to stable manifolds.

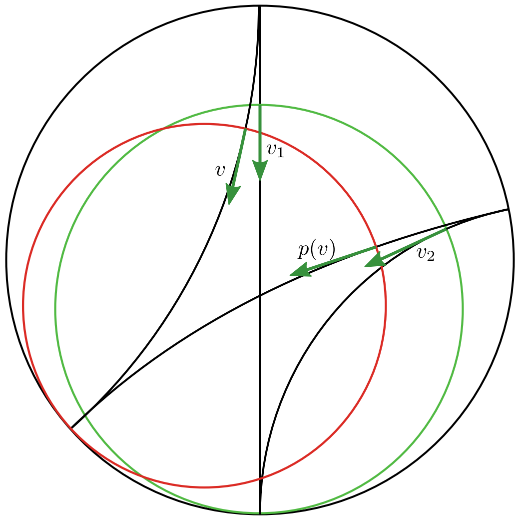

Figure 4.3: The Poincaré map p of the foliation by stable manifolds on the disk. In green we represent (the projection of) the stable manifold of v_1 and in red the stable manifold of v. Black arcs are geodesics.

We will show that the Jacobians q_k of p_k are uniformly bounded in k, that implies the third condition. Then, the lemma allows to conclude that the pullback by p of the volume measure μ_{L_2} is absolutely continuous with respect to μ_{L_1} and that is equivalent to the existence and boundedness of the Jacobian q.

We can reduce the problem to only two cases. We consider the weak unstable manifolds at points v_1 and v_2, that are transversals of the foliation W^s. Let p_{uo} be the Poincaré map between the transversals $W^{uo}(v_1)$ and $W^{uo}(v_2)$ and, for $i = 1, 2$, let p_{L_i} be the Poincaré map between L_i and $W^{uo}(v_i)$. Then we can write the decomposition

$$p = p_{L_2}^{-1} \circ p_{uo} \circ p_{L_1}.$$

So it will be sufficient to prove the existence and boundedness of Jacobians of maps between two unstable manifolds and between every transversal on a point and the unstable manifold at the same point.

For the first case suppose we have $L_i = W^{so}(v_i)$, the maps p, p_k and the Jacobians q, q_k as before. Let $P(v_2) : T^1M \to T^1M$ be the map that at each vector $v \in T^1M$ associates the unique vector with basepoint $\pi(v)$ that is in the weak stable manifold $W^{so}(v_2)$ of v_2. Then the Poincaré map of the foliation by vectors normal to spheres of radius k is decomposed as

$$p_k = g_{-k} \circ P(v_2) \circ g_k$$

if we restrict the geodesic flow g_k to L_1. In Figure 4.4, we represent schematically the situation.

We restrict the map $P(v_2) : g_k(L_1) \to g_k(L_2)$ and denote by J_0 its Jacobian, that exists because the map is differentiable. Notice that $g_k(L_i) = L_i$ because $L_i = W^{so}(v_i)$, so J_0 does not depend on k. We set the notations $T_j^1(v) = T_{g_j(v)}g_j(L_1)$ and $T_j^2(v) = T_{g_j(p_k(v))}g_j(L_2)$, for v in L_1 and $0 \leq j \leq k-1$. We can write the Jacobian of the Poincaré map p_k as a product of Jacobians of the time 1 flow g_1 and J_0.

$$q_k = \prod_{j=0}^{k-1} (J_j^2)^{-1} \cdot J_0 \cdot \prod_{j=0}^{k-1} J_j^1, \tag{4.10}$$

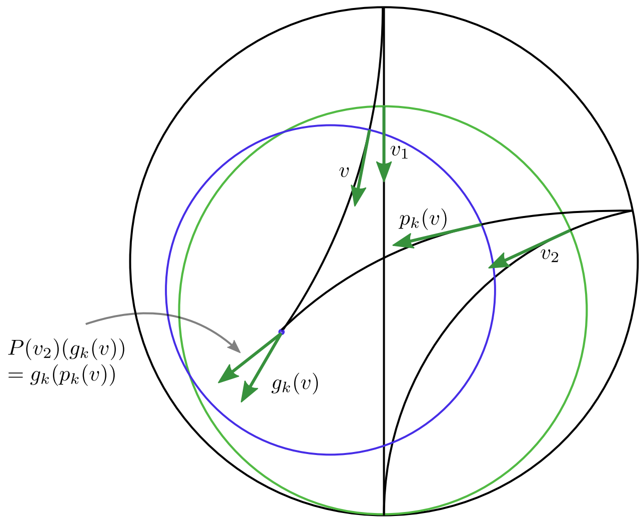

Figure 4.4: The Poincaré map p_k of the foliation by inward vectors to spheres of radius k. In green we represent the stable manifold of v_1 and in blue the sphere of radius k such that v is normal inward.

$$\text{where } J_j^1(v) = |\det(d_{g_j(v)}g_1|_{T_j^1(v)})|,$$
$$\text{and } J_j^2(v) = |\det(d_{g_j(p_k(v))}g_1|_{T_j^2(v)})|.$$

We notice that in fact $T_j^1(v) = E^{uo}(g_j(v))$ and $T_j^2(v) = E^{uo}(g_j(p_k(v)))$. We apply the Hölder continuity of the weak unstable distribution. For all v in L_1,

$$D(T_j^1(v), T_j^2(v)) \leq A d_1(g_j(v), g_j(p_k(v)))^\alpha, \quad k \geq 1,\, 0 \leq j \leq k-1.$$

By Proposition 15, since v and $p_k(v)$ are unit inward vectors to the same sphere, we will have, if k is big enough,

$$d_1(g_j(v), g_j(p_k(v))) \leq Ce^{-aj} d_1(v, p_k(v)), \quad 0 \leq j \leq k-1,$$

for some constants $C, a > 0$. It is clear that the factor $d_1(v, p_k(v))$ is bounded uniformly in v and k. Hence, for k big enough

$$D(T_j^1(v), T_j^2(v)) \leq Be^{-\beta j}, \quad 0 \leq j \leq k-1, \forall v \in L_1,$$

for some constants $B, \beta > 0$.

By hypothesis of regularity, the geodesic flow is C^2, so the tangent map dg_1 is Lipschitz continuous i.e. there is a constant $L > 0$ such that $\left|\!\left|\!\left| d_v g_1 - T_{v,w}^{-1} d_w g_1 \circ T_{v,w} \right|\!\right|\!\right| \leq L\, d_1(v, w)$ for all v, w in T^1M, where $T_{v,w}$ is the parallel transport along the geodesic connecting v to w and $|\!|\!|\cdot|\!|\!|$ is the operator norm of maps between T_vT^1M and $T_{g_1(v)}T^1M$. The Jacobians $J_j^1(v)$ and $J_j^2(v)$ are the determinants in absolute value of the restricted maps $d_{g_j(v)}g_1 : T_j^1(v) \to T_{j+1}^1(v)$ and $d_{g_j(p_k(v))}g_1 : T_j^2(v) \to T_{j+1}^2(v)$, respectively. The directions $T_j^1(v)$ and $T_j^2(v)$ are exponentially close, so

$$\left|J_j^1(v) - J_j^2(v)\right| \leq Ke^{-\beta j}, \quad 0 \leq j \leq k-1, \forall v \in L_1,$$

where K and β are constants. By the compactness of M the uniform norm $\|J_j^i\|$ of the Jacobians is separated away from zero, so we deduce

$$\frac{\|J_j^1\|}{\|J_j^2\|} \leq 1 + K'e^{-\beta j}, \quad 0 \leq j \leq k-1,$$

if k is big. Then, by Equation 4.10, we conclude that $\|q_k\|$ is uniformly bounded in k.

The second case is between a transversal L_1 containing v_1 and the weak unstable manifold $W^{so}(v_1)$ which has the role of L_2. The proof follows the same idea. This time we have the decomposition

$$p_k = g_{-k} \circ P(v_1) \circ g_k.$$

The only difference with the first case is that the restriction $P(v_1) : g_k(L_1) \to g_k(L_2) = W^{so}(v_1)$ depends on k. However, since $T_v L_1$ is transversal to the stable space for v in L_1, we can apply a version of Lemma 9 for the weak unstable spaces and positive time and obtain that

$$D(T_j^1(v), E^{uo}(g_j(v))) \leq K\lambda^j D(T_v L_1, E^{uo}(v)) \tag{4.11}$$

for some constants $K > 1$ and $\lambda \in (0,1)$. We can find constants such that the last inequality is uniform in v. Hence the difference of the new and the old Jacobians decreases exponentially in j in the uniform norm.

The same fact is used to estimate the distance between spaces $T_j^1(v)$ and $T_j^2(v) = E^{uo}(g_j(p_k(v)))$,

$$\begin{aligned} D(T_j^1(v), T_j^2(v)) &\leq D(T_j^1(v), E^{uo}(g_j(v))) + D(E^{uo}(g_j(v)), E^{uo}(g_j(p_k(v)))) \\ &\leq B' e^{-\beta' j}, \quad 0 \leq j \leq k-1, \forall v \in L_1, \end{aligned}$$

for some constant $B', \beta' > 0$. The rest of the proof goes analogously to the first case. □

Putting all the pieces together we obtain the proof of the ergodicity of the geodesic flow.

Theorem 12. *Let M be a C^3 compact Riemannian manifold with negative sectional curvature. Then the geodesic flow g_t on the unitary tangent bundle T^1M is ergodic with respect to the Liouville measure.*

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