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Ergodic properties of horospheres on manifolds without conjugate points

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Abstract

We study the ergodic properties of the horospheres on certain classes of manifolds without conjugate points. Our goal is to generalize several results already known for negatively curved manifolds. We prove that, for a large class of nonpositively curved rank 1 manifolds, certain horospheres are equidistributed under the action of the geodesic flow towards the Bowen-Margulis measure. In the case of nonflat nonpositively curved surfaces, we define a horocyclic flow on the set of horocycles containing a rank 1 vector that is recurrent under the action of the geodesic flow and we prove that this horocyclic flow has a unique invariant probability measure. Finally, we show that any horocyclic flow on a compact higher genus surface without conjugate points and with continuous Green bundles is uniquely ergodic.

Our approach is based on methods specific to geodesic flows such as the boundary at infinity and the construction of the Bowen-Margulis measure via the Patterson-Sullivan theory. The main ingredient in the equidistribution theorem is the mixing of the Bowen-Margulis measure. Regarding the horocyclic flows, our results are obtained thanks to the definition of a uniformly expanding parametrization similar to the one used by B. Marcus in negative curvature.

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Contents

Introduction

The main topic of this thesis is the ergodic theory of horospheres on Riemannian manifolds with a weak form of hyperbolicity. We study two particular classes of manifolds, which are nonpositively curved rank 1 manifolds and compact surfaces without conjugate points. We are interested in properties such as the equidistribution of horospheres and the unique ergodicity of the horospherical foliation. All the results proved here are already known for strict negatively curved manifolds, so our work consisted in extending them for more general classes of manifolds.

The origin of the study of horospheres is closely related to the study of the geodesic flow. As a matter of fact, horocycles are already present in the celebrated proof of E. Hopf of the ergodicity of the Liouville measure with respect to the geodesic flow on hyperbolic surfaces [Hop36].

The first dynamical properties concerning horospheres were obtained by G. A. Hedlund for hyperbolic surfaces [Hed36], namely he proved the minimality of the horocyclic flow on a compact hyperbolic surface. Many years later, H. Furstenberg proved that the horocyclic flow on a compact hyperbolic surface is also uniquely ergodic [Fur73]. B. Marcus generalized this result to variable negative curvature by his own methods.

Theorem. *[Mar75a] Let M be a compact Riemannian surface with strictly negative curvature. Then, any horocyclic flow (stable or unstable) is uniquely ergodic, i.e. there exists a unique invariant Borel probability measure.*

In higher dimension, the fact that a foliation is uniquely ergodic means that there is a unique transverse measure invariant by its holonomy. The unique ergodicity of the horospherical foliation on compact negatively curved manifolds of any dimension follows from the work of R. Bowen and B. Marcus [BM77].

On noncompact hyperbolic or negatively pinched manifolds, we can find both compact and closed embedded horospheres. These provide different transverse measures, so unique ergodicity fails in general. Some authors have studied the transverse invariant measures on hyperbolic manifolds of finite volume [Dan78, Dan81, Rat92] and others on geometrically finite manifolds [Bur90, Rob03]. The most general result is the one given by T. Roblin under the condition that the Bowen-Margulis measure is finite. He shows that, in restriction to a meaningful subset of horospheres, there is a unique transverse measure [Rob03, Théorème 6.4].

The problem of unique ergodicity is naturally related to the equidistribution of horospheres. In the case of the horocyclic flow on a compact surface, this corresponds to the fact that the unique ergodicity is equivalent to the uniform convergence of averages along the flow orbits towards a constant. In the setting of negatively curved manifolds, M. Babillot proved that compact subsets of horospheres get equidistributed when they are pushed by the geodesic flow [Bab02, Theorem 3]. T. Roblin proved a similar result [Rob03, Corollaire 3.2]. In the same work, he also investigated the asymptotics of averages of functions on certain horospherical balls when the radius tends to infinity. Still in negative curvature, B. Schapira studied the equidistribution of horospherical balls towards Gibbs measures for convex-cocompact manifolds [Sch04] and the equidistribution towards the Bowen-Margulis measure for geometrically finite surfaces [Sch05].

Context

The fact that the geodesic flow in negatively pinched curvature is an Anosov flow explains many of the dynamical properties of the horospheres, since they are the stable and the unstable manifolds of the geodesic flow. In this text, we deal with large classes of manifolds defined by geometric constraints whose geodesic flow is not necessarily Anosov. However, these manifolds have some common features with the negatively curved ones, and for this reason we can refer to them as weakly hyperbolic systems. We intend to prove equivalent ergodic properties by generalizing methods which are specific to geodesic flows.

We rely on two different kinds of methods. On the one hand, we can obtain information about the geodesic flow and the horospheres directly from the hypothesis on the manifold, such as curvature restrictions. For general manifolds without conjugate points, it is of special importance the definition of two invariant bundles named after L. Green. On the other hand, the analysis of the global structure of manifolds without conjugate points provides a good theory for studying them. This includes the definition of a boundary at infinity of the universal cover of our manifold and the study of the action of the covering group on these spaces. The Patterson-Sullivan theory then gives the necessary tools in order to study the ergodic properties of the horospheres.

Generally speaking, the lack of hyperbolicity occurs in regions where the stable or unstable horosphere do not coincide with the stable or the unstable manifold, and when the stable horosphere does not intersect transversally with the unstable horosphere. In fact, the two horospheres can intersect in more than one point, this generates a family of non-expanding orbits of the geodesic flow, which is called a strip. Controlling this non-expanding regions is a great step towards obtaining good dynamical results.

Results

Our first result deals with the equidistribution of horospheres under the action of the geodesic flow on nonpositively curved rank 1 manifolds. It is a generalization of the theorem of M. Babillot for negatively curved manifolds [Bab02]. Thanks to the Patterson-Sullivan theory, we can define the Bowen-Margulis measure μ, invariant by the geodesic flow g_t, and a family of measures $\{\mu_H\}_H$ supported on the unstable horospheres of M, uniformly expanded by g_t.

Theorem A. *Let M be a nonpositively curved non-elementary complete connected Riemannian manifold with a closed rank 1 geodesic. Assume that the geodesic flow g_t on the unit tangent bundle T^1M of M is topologically mixing on the set of nonwandering vectors, and that the Bowen-Margulis measure μ is finite. Then, for every unstable horosphere $H \subset T^1M$ containing a g_t-nonwandering vector, every open subset U of H of finite and nonzero μ_H-measure is equidistributed under* *the action of the geodesic flow; i.e. for every bounded and uniformly continuous function f on T^1M, we have*

$$\frac{1}{\mu_H(U)}\int_U f\circ g_t\,\mathrm{d}\mu_H \xrightarrow[t\to+\infty]{} \frac{1}{\mu(T^1M)}\int_{T^1M} f\,\mathrm{d}\mu.$$

The theorem applies to compact nonpositively curved rank 1 manifolds, since all the hypothesis are verified, but also to a large class of noncompact manifolds. The theorem is optimal in the sense that it establishes that an open subset of a horosphere is equidistributed as soon as it has positive measure. It is not difficult to prove that the μ_H-measure of U is positive if and only if it contains a nonwandering rank 1 vector. There exist open sets of horospheres with no rank 1 vectors which are not equidistributed. As an example, we take a surface consisting of a flat cylinder glued to two compact negatively curved ends (Figure 1). All the vertical vectors with base point in a longitudinal segment of the cylinder are in the same unstable horocycle. The set U formed by these vectors has zero μ_H-measure, which is clear from its construction, and vectors on U are periodic, but do not have rank 1. This subset U is not equidistributed in any sense, because when we push it by the geodesic flow it keeps turning around the cylinder. This flat cylinder is an example of what we call a strip.

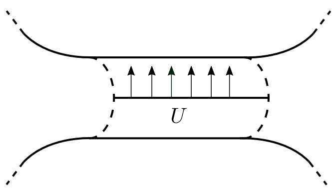


Figure 1: A surface with a flat cylinder

The thesis then focuses on the unique ergodicity of horocyclic flows on weakly hyperbolic surfaces. We restricted our attention to surfaces as a first step to understanding what happens in weakly hyperbolic spaces. Moreover, the methods that we use are rather specific to dimension two.

The unstable horocycles on the unit tangent bundle of a surface without conjugate points form a continuous foliation by Lipschitz curves. A continuous parametrization of these curves is called a horocyclic flow. For example, when the flow is parametrized by the Riemannian length of the horocycles, we say that the horocyclic flow has the Lebesgue parametrization. Since there are many horocyclic flows, in general we should specify the parametrization under study. However, for a continuous flow on a compact space, unique ergodicity is a property which does not depend on the parametrization of the flow. So, for compact surfaces, it is enough to prove or disprove the unique ergodicity for a given parametrization.

We will use a method based on the work of B. Marcus for compact manifolds with negative curvature. The key idea in his proof is the definition of parametrization uniformly expanded by the geodesic flow and called the Margulis parametrization. This gives an explicit description of Birkhoff averages on horocycles when they are pushed by the geodesic flow. Then the equidistribution of horocycles under the action of the geodesic flow implies the pointwise convergence of Birkhoff averages towards a constant, and thus the unique ergodicity of the horocyclic flow.

The uniformly expanding horocyclic flow is obtained by parametrizing the horocycles by the measures μ_H supported on the horocycles. For compact negatively curved surfaces, which is a particular case of manifolds with Anosov geodesic flows, these measures were defined by G. A. Margulis [Mar70]. We also remark that the Margulis and the Lebesgue parametrizations coincide in constant curvature.

Beyond the strict negative curvature case, it is not possible to define the Margulis parametrization in general. This is because of the presence of strips, which yield open subsets of horocycles with zero μ_H-measure as we see in the example of Figure 1.

In the next result, we get around this problem by restricting the domain of definition of the horocyclic flow. On a nonpositively curved surface, a vector has rank 1 if the geodesic tangent to this vector crosses a region with strictly negative curvature. We consider the subset Σ_0 of vectors whose unstable horosphere contains a rank 1 vector recurrent by the geodesic flow. This set contains no strips of non-expanding vectors. We define a horocyclic flow on Σ_0 using the measures on the horocycles and we show that this flow is uniquely ergodic.

Theorem B. *Let M be an orientable rank 1 complete connected C^∞ Riemannian surface with nonpositive curvature. Assume that every vector v in the unit tangent bundle T^1M is nonwandering by the geodesic flow g_t and that the Bowen-Margulis measure μ is finite. Let Σ_0 the set of vectors v whose unstable horocycle contains a rank 1 vector recurrent under the geodesic flow and let h_s be the Margulis horocyclic flow defined on Σ_0. Then every finite Borel measure on Σ_0 invariant under the horocyclic flow h_s is a constant multiple of the Bowen-Margulis measure $\mu|_{\Sigma_0}$ restricted to Σ_0.*

Under the hypothesis of the theorem, the set Σ_0 has full Bowen-Margulis measure and is G_δ-dense in the unit tangent bundle. We notice that the hypothesis requiring that every vector in T^1M is nonwandering by the geodesic flow is satisfied if M has finite Riemannian volume. Moreover, nonflat compact surfaces with nonpositive curvature satisfy all the hypothesis. However, even in this case, the set Σ_0 is not equal to the whole unit tangent bundle. The proof of Theorem B uses the equidistribution of horocycles of Theorem A and a part of the strategy followed by Y. Coudène in [Cou09].

Theorem B gives new information for a large class of nonpositively curved surfaces, but gives no hints about what happens outside the set Σ_0. Our third important result solves this question for a class of compact surfaces larger than nonpositively curved surfaces.

Theorem C. *Let M be an orientable compact C^∞ Riemannian surface without conjugate points, with genus higher than one and continuous Green bundles. Let h_s be any horocyclic flow on the unit tangent bundle T^1M of M. Then there is a unique Borel probability measure on T^1M invariant by the flow h_s.*

The hypothesis on the genus only excludes the case of the flat torus. Indeed, E. Hopf proved that a compact surface without conjugate points is nonflat if and only if it has genus at least two [Hop48]. The continuity of the Green bundles is assumed for technical reasons, and we think that the theorem is probably true without it. In any case, nonpositively curved surfaces, and even surfaces without focal points, have continuous Green bundles. In particular, compact rank 1 nonpositively curved surfaces satisfy the hypothesis of Theorem C, but there are also examples of surfaces satisfying the hypothesis with regions of positive curvature.

We explained before that we cannot define a continuous Margulis parametrization on the whole unit tangent bundle of our manifold. The main obstruction is the presence of strips, which are families of vectors whose orbits by the geodesic flow stay close when viewed in the universal cover of the manifold. On a surface without conjugate points and genus higher than one, a strip appears exactly when there is a nontrivial intersection of a stable horocycle with an unstable one [RR21]. K. Gelfert and R. Ruggiero dealt with these strips by collapsing them, thus obtaining a quotient space with a continuous flow semiconjugated to the geodesic flow. They show that this quotient is a topological 3-manifold when then Green bundles are continuous, and also that the quotient flow is expansive, topologically mixing and has a local product structure [GR20, GR19]. As an application, they deduce the uniqueness of the measure of maximal entropy of the geodesic flow.

It is on this quotient that a uniformly expanding horocyclic flow is defined. A theorem of Y. Coudène, based on B. Marcus's ideas, will guarantee that this flow is uniquely ergodic [Cou09]. Although this horocyclic flow cannot be lifted to the unit tangent bundle of the surface, we succeed in proving that any horocyclic flow above is also uniquely ergodic.

The manuscript also contains two other results that were obtained chronologically between Theorem B and Theorem C. Although both results are now particular cases of Theorem C, we present here the original proofs.

The first result reformulates Theorem B for any parametrization of the horocyclic flow on a compact rank 1 nonpositively curved surface. It is obtained by analyzing how a change of time in the flow changes the invariant measures on the noncompact space Σ_0.

Theorem D. *Let M be an oriented nonflat compact Riemannian surface with nonpositive curvature. Let h_s be a horocyclic flow on the unit tangent bundle T^1M of M and let Σ_0 denote the set of unstable horocycles having a rank 1 vector recurrent under the action of the geodesic flow. Then there is a unique Borel probability measure on T^1M invariant by the flow h_s giving full measure to Σ_0.*

The second result deals with nonpositively curved compact surfaces without strips. The flat strip Theorem ensures that every strip in a nonpositively curved manifold is flat i. e. an isometric immersion of an Euclidean strip. This class of surfaces is interesting from the point of view of generic measures invariant by the geodesic flow. In fact, on a compact nonpositively curved surface, ergodicity is generic in the set of probability measures invariant by the geodesic flow if and only if the surface has no flat strips [CS14]. If there are no strips, the Margulis parametrization of the horocycles is still defined. On these surfaces, we prove the best result one could expect by adapting Coudène's argument in [Cou09].

Theorem E. *Let M be an orientable nonpositively curved compact surface without flat strips and let h_s be any horocyclic flow on T^1M. Then the flow h_s is uniquely ergodic.*

Organization

The manuscript is divided in two parts. The first part is a detailed description of the main geometric tools available in the theory of geodesic flows on manifolds without conjugate points. The second part is devoted to the study of the ergodic properties of the horospheres, so it includes all our results.

The first part achieves a double task. One the one hand, it is intended to be an introduction to the geometric theory of geodesic flows. It constitutes sufficient preparation for a reader with little knowledge of hyperbolic or Riemannian geometry. On the other hand, it summarizes the work of many authors on different aspects of the theory of weakly hyperbolic manifolds. It is a bit more than a compilation of results, since we include the proofs of a great number of results. A certain amount of the proofs needed to be adapted to our context, or we had to clarify its validity in a more general setting. In particular, we think that our treatment of the boundary at infinity is new in the literature.

Part I has two chapters. In Chapter 1, we define the geodesic flow and we cover some basic Riemannian geometry such as the exponential map, Jacobi fields and the curvature tensor. We also define the Green bundles, which are important towards the end of the manuscript. In Chapter 2, we introduce a few more objects such as the boundary at infinity, Busemann functions and horospheres, thanks to which it is possible to understand the structure of a large class of manifolds without conjugate points.

Part II contains all the equidistribution and unique ergodicity results explained above. In a first lecture, the reader who is already familiar with geodesic flows, or who is ready to accept a certain amount of facts as true, can skip directly to this part.

Part II has four chapters. In Chapter 3, we explain the relation between the limit set and nonwandering vectors under the geodesic flow. Then we define the Patterson-Sullivan measure, and we construct the Bowen-Margulis measure for the two situations under investigation: nonpositively curved rank 1 manifolds and compact surfaces without conjugate points. We finally define the measures μ_H on the horospheres, which will be essential in the sequel. In Chapter 4, we establish an equivalence concerning the topological mixing of the geodesic flow and the mixing of the Bowen-Margulis measure and we prove Theorem A about the equidistribution of horospheres. Theorems B, D and E concerning the unique ergodicity of the horocyclic flow on nonpositively curved surfaces are proved in Chapter 5. Finally, Chapter 6 contains a brief explanation of the quotient by strips and the proof of Theorem C.

The results of this thesis constitute three articles. The first [BC21a] is published in Ergodic Theory and Dynamical Systems, the second [BC21b] is submitted for publication and the third [BC22] will be submitted for publication soon.

Résumé en français

L'objectif principal de cette thèse est l'étude des propriétés ergodiques des horosphères dans les variétés riemanniennes présentant une forme faible d'hyperbolicité. Nous visons spécifiquement deux classes de variétés, qui sont les variétés de rang 1 à courbure négative ou nulle et les surfaces compactes sans points conjugués. Nous nous intéressons aux propriétés telles que l'équidistribution des horosphères et l'unique ergodicité du flot horocyclique. Tous les résultats ici présents sont déjà connus en courbure strictement négative, donc notre travail a consisté à généraliser ces résultats.

Le cadre classique pour l'étude des horosphères est celui de la géométrie hyperbolique. Le flot géodésique de ces variétés, aussi bien que des variétés à courbure négative pincée, est un exemple de flot d'Anosov et les horosphères ne sont rien d'autre que les variétés stables et instables de ce flot. Un grand nombre de propriétés ergodiques des horosphères découlent alors directement de ce fait. Dans ce texte, nous faisons face à des systèmes définis par des contraintes géométriques qui n'auront pas la propriété d'Anosov, mais qui rassembleront encore dans certains aspects aux flots géodésiques en courbure négative.

Ainsi, on sera amené à l'emploi d'outils liés à la géométrie que nous décrivons brièvement maintenant. D'une part, les hypothèses sur notre espace, comme par exemple sur la courbure, imposeront des restrictions au comportement des géodésiques, qui se transporteront à la dynamique du flot géodésique. La définition des fibrés de Green sur notre variété en absence de points conjugués sera aussi essentielle par la suite. D'autre part, l'analyse de la structure globale de notre espace à partir du bord à l'infini et l'étude de l'action du groupe de revêtement sur ce bord sont très importantes. Ceci permet d'établir une théorie de Patterson-Sullivan pour les espaces étudiés, qui fournit les éléments nécessaires pour l'obtention de propriétés ergodiques des horosphères.

En règle générale, dans les variétés sans points conjugués, on observera des phénomènes non hyperboliques comme des horosphères qui ne sont pas dilatantes ou contractantes ou le fait qu'une horosphère stable intersecte une horosphère instable de manière non transversale. En fait, ces dernières horosphères pourront s'intersecter en plusieurs points, ce qui va engendrer une bande non expansive d'orbites du flot géodésique. Le contrôle de ces bandes est un premier pas important vers la compréhension des propriétés dynamique du flot géodésique et des horosphères.

Décrivons maintenant en détail nos résultats. Nous commençons par présenter un résultat d'équidistribution des horosphères sous l'action du flot géodésique. Nous aurons défini, grâce aux mesures de Patterson-Sullivan, une mesure μ invariante par le flot géodésique, appelée mesure de Bowen Margulis, et une famille de mesures $\{\mu_H\}_H$ supportées sur les horospheres instables de M qui sont uni-

formément dilatées par le flot géodésique. On considère les moyennes d'une fonction par rapport à une mesure μ_H sur un ouvert dans une horosphère instable poussées par le flot géodésique en un temps long.

Théorème A. *Soit M une variété Riemannienne C^∞ complète et non-élémentaire, à courbure négative ou nulle et qui contient une géodésique fermée de rang 1. Supposons que le flot géodésique g_t sur le fibré tangent unitaire T^1M de M est topologiquement mélangeant sur l'ensemble non-errant de g_t, et que la mesure de Bowen-Margulis μ est finie. Alors, pour toute horosphère instable $H \subset T^1M$ contenant un vecteur non-errant par g_t, tout ouvert U de H avec mesure μ_H finie et non nulle est équidistribué sous l'action du flot géodésique, autrement dit, pour toute fonction f sur T^1M uniformément continue et bornée, on a*

$$\frac{1}{\mu_H(U)} \int_U f \circ g_t \, \mathrm{d}\mu_H \xrightarrow[t \to +\infty]{} \frac{1}{\mu(T^1M)} \int_{T^1M} f \, \mathrm{d}\mu.$$

Ce résultat généralise un théorème de M. Babillot en courbure négative [Bab02]. Les variétés compactes de rang 1 à courbure négative ou nulle, ainsi qu'un grand nombre de variétés non compactes, vérifient les hypothèses du théorème. Ce théorème est optimal dans le sens où il établit qu'un ouvert d'une horosphère est équidistribué dès qu'il a une mesure positive. La mesure μ_H de U est positive si et seulement si il contient un vecteur non errant de rang 1. Il existe des parties ouvertes d'une horosphère qui n'ont pas de vecteurs de rang 1 et qui ne s'équidistribuent pas. Par exemple, on prend une surface formée par un cylindre plat collé à deux bouts compacts à courbure négative (Figure 1). Tous les vecteurs verticaux dont le point base est dans un segment longitudinal du cylindre sont dans le même horocycle instable. L'ensemble U formé par ces vecteurs est de mesure μ_H nulle, ce qui ressort clairement de sa construction, et les vecteurs sur U sont périodiques, mais ne sont pas de rang 1. Ce sous-ensemble U n'est en aucun cas équidistribué, car lorsque nous le poussons par le flot géodésique, il tourne autour du cylindre. Ce cylindre plat est un exemple de ce que nous appelons une bande.

La thèse porte ensuite sur l'unique ergodicité des flots horocycliques sur des surfaces faiblement hyperboliques. Les horocycles instables sur le fibré unitaire tangent d'une surface sans points conjugués forment un feuilletage continu par courbes lipschitziennes. Un flot horocyclique est une paramétrisation continue de ces courbes. Par exemple, lorsque le flot est paramétré par la longueur d'arc des horocycles, on dit que le flot horocyclique a la paramétrisation de Lebesgue. Comme il existe de nombreux flots horocycliques, il convient en général de préciser la paramétrisation étudiée. Cependant, pour un flot continu sur un espace compact, l'ergodicité unique est une propriété qui ne dépend pas de la paramétrisation. En conséquence, pour les surfaces compactes, il suffit de prouver ou d'infirmer l'unique ergodicité pour une paramétrisation donnée.

Nous utiliserons une méthode basée sur les travaux de B. Marcus pour les variétés compactes à courbure négative. L'idée clé de sa preuve est la définition d'une paramétrisation uniformément dilatée par le flot géodésique, qu'on appelera paramétrisation de Margulis. Cela donne une description explicite des moyennes de Birkhoff sur les horocycles lorsqu'ils sont poussés par le flot géodésique. Alors l'équidistribution des horocycles sous l'action du flot géodésique implique la convergence simple des moyennes de Birkhoff vers une constante, donc l'unique ergodicité du flot horocyclique.

Le flot horocyclique uniformément dilaté est obtenu en paramétrant les horocycles par les mesures μ_H supportées sur eux. Pour les surfaces compactes à courbure négative, qui sont un cas particulier des variétés avec flots géodésiques d'Anosov, ces mesures ont été définies par G. A. Margulis [Mar70]. On remarque aussi que la paramétrisation de Margulis et celle de Lebesgue coïncident en courbure constante.

Au-delà du cas de courbure négative, il n'est pas toujours possible de définir la paramétrisation de Margulis. Cela est dû à la présence de bandes, qui donnent lieu à des ouverts dans les horocycles de mesure μ_H nulle, comme nous l'avions vu dans l'exemple de la Figure 1.

Dans le résultat suivant, nous contournons ce problème en restreignant le domaine de définition du flot horocyclique. Sur une surface à courbure négative ou nulle, un vecteur est de rang 1 si la géodésique engendré par ce vecteur traverse une région à courbure strictement négative. On considérera le sous-ensemble Σ_0 formé par les vecteurs dont l'horosphère instable contient un vecteur de rang 1 récurrent par le flot géodésique. Cet ensemble ne contient aucune bande non expansive de vecteurs. Nous définirons un flot horocyclique sur Σ_0 à partir des mesures sur les horocycles et nous montrerons que ce flot est uniquement ergodique.

Théorème B. *Soit M une surface Riemannienne C^∞ complète, connexe, orientable et de rang 1 à courbure négative ou nulle. Supposons que tous les vecteurs v du fibré tangent unitaire T^1M sont non errants par le flot géodésique g_t et que la mesure de Bowen-Margulis μ est finie. Soit Σ_0 l'ensemble des vecteurs v dont l'horocycle instable contient un vecteur de rang 1 récurrent par le flot géodésique et soit h_s le flot horocyclique de Margulis défini sur Σ_0. Alors, toute mesure borélienne finie sur Σ_0 invariante par le flot horocyclique h_s est, à une constante multiplicative près, la mesure de Bowen-Margulis $\mu|_{\Sigma_0}$ restreinte à Σ_0.*

Sous les hypothèses du théorème, l'ensemble Σ_0 est de mesure de Bowen-Margulis totale et il est G_δ-dense dans le fibré unitaire tangent. Nous remarquons que tous les vecteurs de T^1M sont non errants par le flot géodésique si M est de volume riemannien fini. De plus, les surfaces compactes non plates à courbure négative ou nulle satisfont toutes les hypothèses. Cependant, même dans ce cas, l'ensemble Σ_0 n'est pas égal au fibré unitaire tangent entier. La preuve du théorème B utilise l'équidistribution des horocycles du théorème A et une partie de la stratégie suivie par Y. Coudène dans [Cou09].

Le théorème B donne de nouvelles informations pour une large classe de surfaces à courbure négative ou nulle, mais ne donne aucune indication sur ce qui se passe en dehors de l'ensemble Σ_0. Notre troisième résultat résout cette question pour une classe de surfaces compactes qui inclut les surfaces à courbure négative ou nulle.

Théorème C. *Soit M une surface Riemannienne C^∞ compacte orientable, de genre supérieur à un, sans points conjugués et avec fibrés de Green continus. Soit h_s un flot horocyclique quelconque du fibré tangent unitaire T^1M de M. Alors, il y a une unique mesure de probabilité borélienne sur T^1M invariante par le flot h_s.*

On suppose que M est de genre supérieur à un afin d'exclure le cas du tore plat. En effet, E. Hopf a prouvé qu'une surface compacte sans points conjugués est non plate si et seulement si elle a genre au moins deux [Hop48]. On suppose la continuité des fibrés de Green pour des raisons techniques, mais nous pensons que le théorème est probablement vrai sans cette hypothèse. De toute façon, les surfaces à courbure négative ou nulle, et même les surfaces sans points focaux, ont des fibrés de Green continus. En particulier, les surfaces compactes de rang 1 à courbure négative ou nulle satisfont les hypothèses du théorème C, mais il existe aussi des exemples de surfaces satisfaisant les hypothèses avec des régions à courbure positive.

Pour montrer le théorème C, nous utiliserons un flot expansif et avec structure de produit local associé au flot géodésique défini par K. Gelfert et R. Ruggiero [GR20, GR19]. C'est sur le nouveau espace de phases qu'on définit un flot horocyclique uniformément dilaté. Un théorème d'Y. Coudène, basé sur les idées de B. Marcus, garantira que ce flot est uniquement ergodique [Cou09]. Bien que ce flot horocyclique ne puisse pas être relevé directement au fibré unitaire tangent de la surface, nous réussissons à prouver que tout flot horocyclique du fibré tangent unitaire est aussi uniquement ergodique.

Part I

Geometry of weakly hyperbolic spaces

Chapter 1

Reminders on Riemannian geometry

This chapter introduces several differentiable tools for the study of geodesic flows, paying special attention to manifolds without conjugate points or satisfying stronger assumptions. It starts from elementary Riemannian geometry, describing the double tangent bundle of a manifold and the action of the differential of the geodesic flow. We recall the relation between the curvature of a manifold and Jacobi fields, which has consequences on the dynamics of the geodesic flow. The final goal of the chapter is to define two subbundles invariant by the geodesic flow, named after L. Green, which will be present in the rest of the manuscript.

1.1 Geodesic flow

In the whole text we will work in the following setting:

Standing assumption: M is a C^∞ complete Riemannian manifold of dimension $n \geq 2$.

Let us detail these hypotheses. A Riemannian manifold is a differentiable manifold M together with a positive definite symmetric 2-covariant tensor g on M. Both the manifold M and the Riemannian metric g are assumed to be C^∞. The projection from the tangent bundle TM of M to the base M is denoted by π. The metric g induces a scalar product on each tangent space T_xM, $x \in M$ that will be denoted by $\langle \cdot, \cdot \rangle$, and also a norm $\|\cdot\|$. The unit tangent bundle T^1M is the subbundle of TM formed by the vectors with norm 1, the projection from T^1M to M is also denoted by π.

The length of a smooth curve $c : [a, b] \to M$ is defined as

$$l(c) = \int_a^b \|\dot{c}(t)\| \, dt.$$

There is a natural Riemannian distance d on M, which for two points $p, q \in M$ can be defined as

$$d(p, q) = \inf\{l(c) \mid c \text{ is a piecewise } C^1 \text{ curve joining } p \text{ to } q\}.$$

A differentiable curve $c : (a, b) \to M$ is a geodesic if it satisfies the second order equation $\nabla_{\dot{c}}\dot{c} = 0$, were ∇ is the Levi-Civita connection associated to the metric

g. We recall that geodesics locally minimize length, and that if a curve minimizes length, it is a geodesic up to reparametrization by constant speed. For a vector $v \in TM$, we denote by c_v the unique geodesic such that $c_v(0) = \pi(v)$ and $\dot{c}_v(0) = v$. The Hopf-Rinow theorem says that (M, d) is complete as a metric space if and only if the geodesics are defined for all time. We will assume that M is always complete and consider the geodesics c_v defined on $\mathbb{R}$. In this situation we can define the geodesic flow as follows.

Definition 1.1.1. The geodesic flow on T^1M is the family of maps $g_t : T^1M \to T^1M$ for $t \in \mathbb{R}$ defined by $g_t(v) = \dot{c}_v(t)$, where $v \in T^1M$.

In order to study the geodesic flow more in depth, let us first describe the structure of T^1M. Since TM and T^1M are manifolds themselves, we can consider the respective tangent bundles TTM and TT^1M. The fibers of these bundles are described by the following proposition.

Proposition 1.1.1. *Let $v \in TM$. For each $Z \in T_vTM$, let $V_Z : (-\varepsilon, \varepsilon) \to TM$ be a curve passing through v with direction Z, $V_Z(0) = v$, $\dot{V}_Z(0) = Z$, so $c_Z = \pi \circ V_Z$ is a curve on M and let $\frac{DV_Z}{dt}$ denote the covariant derivative (V_Z is thought as a vector field along c_Z). Then the map*

$$\begin{array}{ccc} T_vTM & \longrightarrow & T_{\pi(v)}M \oplus T_{\pi(v)}M \\ Z & \longmapsto & \left(\dot{c_Z}(0),\, \frac{DV_Z}{dt}(0)\right) \end{array}$$

is a isomorphism of vector spaces. Since T^1M is a submanifold of TM, for $v \in T^1M$, T_vT^1M is a subspace of T_vTM, which through the previous isomorphism corresponds to $T_{\pi(v)}M \oplus v^\perp$.

Thanks to this decomposition we define a natural Riemannian metric g^S on TM called the Sasaki metric. This metric is defined for two vectors $Z_1 = (X_1, Y_1), Z_2 = (X_2, Y_2) \in T_vTM$ in the decomposition provided by Proposition 1.1.1 by

$$g^S_v(Z_1, Z_2) = g_{\pi(v)}(X_1, X_2) + g_{\pi(v)}(Y_1, Y_2).$$

As a Riemannian metric on the unit tangent bundle T^1M we consider the restriction of g^S, which is also called Sasaki metric.

1.2 Curvature and Jacobi fields

Now we introduce the curvature tensor and the Jacobi equation. Let ∇ denote the Levi-Civita connection of M. The curvature tensor R of the manifold M is the $(1,3)$-tensor defined by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z,$$

where X, Y, Z are three vector fields on M. The value of $R(X,Y)Z$ at a point $p \in M$ only depends of the values of X, Y and Z at p. Given two linearly independent vectors $v, w \in T_pM$, we write

$$K_p(v,w) = \frac{g(R(v,w)v, w)}{g(v,v)g(w,w) - g(v,w)^2}.$$

One can check that the above quantity does not depend on the choice of the basis of the space $\langle v, w\rangle \subset T_pM$. So it defines a function K on the set of tangent planes of M. This function is called the sectional curvature of M.

Let I be an interval of $\mathbb{R}$ and $c : I \to M$ a geodesic. A variation of the geodesic c is a differentiable map $f : (-\varepsilon, \varepsilon) \times I \to M$ such that for each $s \in (-\varepsilon, \varepsilon)$, $t \mapsto f(s,t)$ is a geodesic.

A vector field V along a geodesic $c : I \to M$ is a differentiable map $V : I \to TM$ such that $V(t) \in T_{c(t)}M$ for all $t \in I$. We denote $\frac{DV}{dt}$, or V' for short, the covariant derivative of V with respect to the time t, that is $DV/dt(t) = (\nabla_{\dot{c}}\tilde{V})_{c(t)}$, where $\tilde{V}$ is an extension of V to a neighborhood. A variation f of geodesic c comes with a vector field $\frac{\partial f}{\partial s}$ in the direction of the variation, formally defined as

$$\frac{\partial f}{\partial s}(s,t) = d_{(s,t)}f(\frac{\partial}{\partial s}).$$

We first introduce Jacobi fields as solutions of a certain equation.

Definition 1.2.1. A Jacobi field along a geodesic c is a vector field $J : I \to TM$ along c which satisfies the equation

$$J'' + R(\dot{c}, J)\dot{c} = 0.$$

Since the Jacobi equation is a second order linear differential equation, its solutions are uniquely determined by the initial conditions $X = J(0) \in T_{c(0)}M$ and $Y = J'(0) \in T_{c(0)}M$ $(0 \in I)$. The space of Jacobi fields along a geodesic is thus a vector space of dimension $2n$. There are two trivial solutions $J(t) = \dot{c}(t)$ and $t\dot{c}(t)$ which correspond to the initial conditions $(X,Y) = (\dot{c}(0), 0)$ and $(0, \dot{c}(t))$. We observe that these solutions are tangent to $\dot{c}$ all the time. The Jacobi fields with initial conditions perpendicular to $\dot{c}(0)$ (i.e. $g(X, \dot{c}(0)) = g(Y, \dot{c}(0)) = 0$) form a subspace of dimension $2n-2$. These Jacobi fields will be called perpendicular or orthogonal, and they are exactly the ones which are perpendicular to $\dot{c}(t)$ for all t, because of the formula

$$g(J(t), \dot{c}(t)) = g(J(0), \dot{c}(0)) + tg(J'(0), \dot{c}(0)). \tag{1.1}$$

We can now characterize Jacobi fields as infinitesimal variations of geodesics.

Proposition 1.2.1. *For a variation f of a geodesic c the vector field $\frac{\partial f}{\partial s}|_{s=0}$ is a Jacobi field on c. Conversely, any Jacobi field can be integrated into a variation of geodesics.*

Before going further in the study of Jacobi fields, let us explain how they are related to geodesic flow.

Proposition 1.2.2. *Let $t \in \mathbb{R}$ and $v \in TM$. The differential of the geodesic flow $d_vg_t : T_{\pi(v)}M \oplus T_{\pi(v)}M \to T_{\pi(g_tv)}M \oplus T_{\pi(g_tv)}M$ at v is given by*

$$d_vg_t(X,Y) = (J(t), J'(t)),$$

where J is the unique Jacobi field along γ_v such that $J(0) = X$ and $J'(0) = Y$.

Proof. Let $(X, Y) \in T_{\pi(v)}M \oplus T_{\pi(v)}M$. Consider a smooth curve $s \mapsto V(s)$ on TM with $c = \pi \circ V$ such that $\dot{c}(0) = X$ and $\frac{DV}{ds}(0) = Y$. Then $g_t \circ V$ is a curve passing through $g_t v$ and, by definition, with direction $d_v g_t(X, Y)$. In the decomposition of the double tangent bundle, we have

$$d_v g_t(X, Y) = \left(\frac{d}{ds}\bigg|_{s=0} (\pi \circ g_t \circ V)(s), \frac{D(g_t \circ V)}{ds}(0) \right).$$

We consider the geodesic variation

$$f(s, t) = \gamma_{V(s)}(t) = \pi(g_t(V(s)))$$

and observe that $\frac{\partial f}{\partial t}(s, t) = g_t(V(s))$. So we obtain, for all $t \in \mathbb{R}$,

$$d_v g_t(X, Y) = \left(\frac{\partial f}{\partial s}(0, t), \left(\frac{D}{ds} \frac{\partial f}{\partial t} \right)(0, t) \right) = \left(\frac{\partial f}{\partial s}(0, t), \left(\frac{D}{dt} \frac{\partial f}{\partial s} \right)(0, t) \right).$$

We know that $J(t) = \frac{\partial f}{\partial s}(0, t)$ is a Jacobi field along γ_v and the last term is exactly $(J(t), J'(t))$. For $t = 0$ we see that $(J(0), J'(0)) = (X, Y)$. □

We can now relate the growth of the differential of the geodesic flow in the Sasaki metric with the growth of Jacobi fields and their derivatives thanks to the formula

$$||d_v g_t(J(0), J'(0))|| = \sqrt{||J(t)||^2 + ||J'(t)||^2}. \tag{1.2}$$

Later we will look closer to the action of the differential of geodesic flow and define some special invariant subspaces of the unit tangent space. For now, just observe that the directions of the geodesic flow are given by

$$G(v) = \frac{d}{dt}|_{t=0} g_t(v) = (v, 0),$$

and that the orthogonal spaces in TT^1M are

$$G(v)^\perp = v^\perp \oplus v^\perp,$$

which are invariant by dg_t thanks to Equation 1.1 and the fact that $g(J'(t), \dot{c}(t))$ is constant when J is a Jacobi field.

Cartan-Hadamard theorem

The exponential map at the point $p \in M$ is the map $\exp_p : T_pM \to M$ defined by

$$\exp_p(v) = c_v(1), \quad v \in T_pM.$$

This map is defined everywhere on T_pM because M is complete and is surjective by the Hopf-Rinow theorem. Given a curve $V : (-\varepsilon, \varepsilon) \to T_pM$, we can consider the following variation of geodesics:

$$f(s, t) = \exp_p(tV(s)).$$

Lemma 1.2.3. *Let $p \in M$, $v \in T_pM$ and $w \in T_vT_pM \equiv T_pM$. The differential of the exponential map is*

$$d_v \exp_p(w) = J(1),$$

where J is the Jacobi field on c_v with $J(0) = 0$ and $J'(0) = w$.

Thanks to this explicit formula for the differential of the exponential map, we see that the critical points of $\exp_p$ are those v for which there exists a nonzero Jacobi field J on c_v which vanishes at 0 and at another point $t \neq 0$. When this happens, the point $c_v(t)$ with $J(t) = 0$ is said to be conjugate to p.

Definition 1.2.2. A manifold M has no conjugate points if, for each $p \in M$, p has no conjugate points.

Proposition 1.2.4. *If M is a manifold without conjugate points, the exponential map $\exp_p$ at any point p is a universal covering map.*

Proof. Let us show that $\exp_p$ is a covering map by proving the path lifting property. First of all, we observe that $\exp_p$ is a local diffeomorphism by the remark made just after 1.2.3.

Then, we lift the metric of M to T_pM. The geodesics passing through 0 with this new metric are straight lines. The Hopf-Rinow theorem implies that T_pM with the distance given by this new metric is complete, since geodesics from 0 are defined for all time.

The following lemma, together with the fact that T_pM is simply connected, concludes the proof. In our, case f is a local isometry.

Lemma 1.2.5. *[dC92, Lemma 3.3, Chap. 7] Let N be complete Riemannian manifold and $f : N \to M$ a local diffeomorphism with $\|df\| \geq 1$. Then f is a covering map.*

$\square$

For a manifold M, we denote by $\tilde{M}$ a universal cover equipped with the pullback metric and $\Pi : \tilde{M} \to M$ the covering map. If M has no conjugate points, the space $\tilde{M}$ can be thought as an open ball of the same dimension as M with a certain metric, and $\tilde{M}$ has no conjugate points either.

Proposition 1.2.6. *If M has no conjugate points, the geodesics of $\tilde{M}$ are globally minimizing.*

Proof. For any point $p \in \tilde{M}$, the exponential map $\exp_p : T_p\tilde{M} \to \tilde{M}$ is a bijection, since $\tilde{M}$ is already simply connected. In fact, the map $\exp_p$ is a diffeomorphism. This implies that, given another point $q \in \tilde{M}$, there is a unique geodesic c joining p to q up to reparametrization. Since the distance between two points is always attained by the length of a geodesic, we deduce that geodesics minimize length globally. $\square$

Throughout this manuscript, we will consider curvature bounds of the type $K \leq \kappa$ or $K \geq \kappa$, $\kappa \in \mathbb{R}$, where K is the sectional curvature of M and the inequalities are meant to hold for all the planes over all the points of M. One important case are manifolds with nonpositive curvature, that is, $K \leq 0$.

Before going further, we introduce another class of manifolds, the so-called manifolds without focal points. We have not worked directly with these kind of manifolds, but the concept it will be useful to understand some parts of the theory.

Definition 1.2.3. Let $c : I \to M$ be a geodesic starting at a point $c(0) = p$. For $t \in I$, the point $c(t)$, $t \in I$ is said to be focal to p if there exists a nonzero Jacobi field $J : I \to TM$ along c such that $J(0) = 0$ and $(\|J\|^2)'(t) = 0$. The manifold M has no focal points if for every $p \in M$ there are no focal points to p.

In the rest of this section, we explore further the behaviour of Jacobi fields and the Riemannian distance between geodesics for nonpositively curved manifolds, manifolds without focal points and manifolds without conjugate points. The following proposition gives a characterization of each of the three classes of manifolds in consideration in terms of the growth of Jacobi fields.

Proposition 1.2.7. *Let M be a complete Riemannian manifold.*

1. *M has nonpositive curvature if and only if $(\|J\|^2)'' \geq 0$ for every Jacobi field J along every geodesic.*
2. *M has no focal points if and only if $(\|J\|^2)'(t) > 0$ for all $t > 0$ for every nonzero Jacobi field J vanishing at 0.*
3. *M has no conjugate points if and only if $\|J\|^2$ vanishes at most once for every nonzero Jacobi field J along every geodesic.*

Proof. 1. Let J be a Jacobi field along c. We compute

$$(\|J\|^2)'' = \langle J, J\rangle'' = 2\langle J', J\rangle' = 2\langle J', J'\rangle + 2\langle J'', J\rangle = 2\langle J', J'\rangle - 2\langle R(\dot{c}, J)\dot{c}, J\rangle.$$

The first term of the last inequality is always nonnegative. The second is either 0 if $\dot{c}$ and J are linearly dependent or equal to $-K_c(\dot{c}, J)(\|\dot{c}\|^2 \|J\|^2 - \langle \dot{c}, J\rangle^2)$ in the other case. If the curvature of M is nonpositive, we see that in both cases $(\|J\|^2)'' \geq 0$.

Conversely, let $v, w \in T_pM$ be linearly independent. Consider the geodesic c through p with direction v, and then the Jacobi field J along c with initial conditions $J(0) = w$ and $J'(0) = 0$. Then, $(\|J\|^2)''(0) = -K_p(v, w)(\|v\|^2 \|w\|^2 - \langle v, w\rangle^2) \geq 0$, so $K_p(v, w) \leq 0$.

2. If M has no focal points, for every nonzero Jacobi field J with $J(0) = 0$, we have $(\|J\|^2)'(t) \neq 0$ for all $t > 0$. Since $\|J\|^2 \geq 0$ and $\|J\|^2(0) = 0$, it must be $(\|J\|^2)'(t) > 0$ for all $t > 0$. The converse is immediate.

3. By definition, two points are conjugate if there exists a nonzero Jacobi field vanishing at both points.

□

The next result now becomes clear.

Corollary 1.2.8. *If M has nonpositive sectional curvature, then it has no focal points. If M has no focal points, then it has no conjugate points.*

We next give a more precise estimate of the growth of Jacobi fields that we will need later.

Proposition 1.2.9. *Assume that the curvature on all the planes tangent to a geodesic $c : I \to M$ is less than a constant $-\kappa \in \mathbb{R}$. Let J be a perpendicular Jacobi field on c. Then, if J does not vanish on I, we have for all $t \in \mathbb{R}$,*

$$\|J\|''(t) \geq \kappa \|J\|(t).$$

If we integrate the conditions on the Jacobi fields, we obtain the next proposition about the distance between geodesics on the universal cover.

Proposition 1.2.10. *Let c_1, $c_2 : [0,1] \to \tilde{M}$ be two geodesics with any speed on the universal cover of a manifold M without conjugate points. If M is nonpositively curved, then for all $t \in [0,1]$,*

$$d(c_1(t), c_2(t)) \leq td(c_1(0), c_2(0)) + (1-t)d(c_1(1), c_2(1)).$$

If M has no focal points, then for all $t \in [0,1]$,

$$d(c_1(t), c_2(t)) \leq d(c_1(0), c_2(0)) + d(c_1(1), c_2(1)).$$

Proof. It is enough to proof each of the statements for two geodesics c_1, c_2 with $c_1(0) = c_2(0) =: p$.

Let $c : [0,1] \to \tilde{M}$ denote the unique geodesic joining $c_1(1)$ to $c_2(1)$. We can write $c(s) = \exp_p(V(s))$ for some curve $V : [0,1] \to T_p\tilde{M}$. We consider the variation of geodesics $f(t,s) = \exp_p(tV(s))$ and the Jacobi field $J_s(t) = \frac{\partial f}{\partial s}(t,s)$ along the geodesic $t \mapsto f(t,s)$. Now, for $t \in [0,1]$

$$d(c_1(t), c_2(t)) \leq \int_0^1 ||\frac{\partial f}{\partial s}(t,s)||\mathrm{d}s = \int_0^1 ||J_s(t)||\mathrm{d}s. \tag{1.3}$$

For each $s \in [0,1]$, J_s is a Jacobi field with $J_s(0) = 0$. If $\tilde{M}$ is nonpositively curved, since $\|J_s\|$ is convex, we have $\|J_s(t)\| \leq t\,\|J_s(1)\|$. If $\tilde{M}$ has no focal points, $\|J_s(t)\| \leq \|J_s(1)\|$. Applying these inequalities to (1.3), since $J_s(1) = \dot{c}(s)$, we get the desired result. □

To finish the section, we explain why focal points receive this name. Let N be a submanifold of M. We denote the normal bundle of N by $N^\perp$. Let $c : \mathbb{R} \to M$ be a unit speed geodesic with $c(0) \in N$ and $\dot{c}(0)$ perpendicular to N. Consider a variation $f : (-\varepsilon, \varepsilon) \times \mathbb{R} \to M$ of the geodesic c such that for all $t \in \mathbb{R}$, $s \in (-\varepsilon, \varepsilon)$,

$$f(s,0) \in N, \quad \frac{\partial f}{\partial t}(s,0) \perp T_{f(s,0)}N. \tag{1.4}$$

An N-Jacobi field along c is a Jacobi field J obtained from a variation of geodesics satisfying (1.4), i.e. $J(t) = \frac{\partial f}{\partial s}(0,t)$.

N-Jacobi fields also admit an infinitesimal characterization. Given a submanifold N, $p \in N$, and a vector $X \in T_pM$, we denote the projection of X to T_pN by X^T. For a perpendicular vector $v \in (T_pN)^\perp$, we consider an extension V of v normal to N on a neighborhood and define the shape operator $S_v : T_pN \to T_pN$ by

$$S_v(X) = -(\nabla_X V)^T.$$

Proposition 1.2.11. *A Jacobi field J along c is an N-Jacobi field if an only if*

$$J(0) \in T_pN, \quad J'(0) + S_{\dot{c}(0)}J(0) \in (T_pN)^\perp. \tag{1.5}$$

We can now define the concept of focal points of a submanifold.

Definition 1.2.4. The point $c(t)$ is focal to the submanifold N if there exists an N-Jacobi field along c such that $J(t) = 0$.

We denote the normal bundle of N by $TN^\perp$. The normal exponential map $\exp^\perp : TN^\perp \to M$ is defined by $\exp(v) = \exp_{\pi(v)}(v)$. We can relate the focal points of N to the singularities of the normal exponential map.

Lemma 1.2.12. *Focal points of a submanifold N are the critical values of the normal exponential map $\exp^\perp : TN^\perp \to M$.*

The concept of focal points along a geodesic corresponds to focal points of geodesic manifolds.

Proposition 1.2.13. *Let c be a unit speed geodesic with $c(0) = p$, and $\dot{c}(0) = v$. Then the following are equivalent:*

1. *There exists a perpendicular Jacobi field J along c such that $J(t) = 0$ and $(\|J\|^2)'(0) = 0$.*
2. *The point $c(t)$ is focal to a totally geodesic manifold N containing p perpendicular to v.*

In particular, $\tilde{M}$ has no focal points if and only if no totally geodesic submanifold has focal points.

Proof. 2. $\Longrightarrow$ 1. Let J be an N-Jacobi field vanishing at $c(t)$. Since N is totally geodesic, the shape operator is identically zero, hence $J'(0) \in (T_pN)^\perp$ and

$$\frac{1}{2}(\|J\|^2)'(0) = \langle J(0), J'(0)\rangle = 0.$$

1. $\Longrightarrow$ 2. Let J be a perpendicular Jacobi field along c such that $(\|J\|^2)'(0) = 0$ and $J(t) = 0$. The manifold N that we consider is the geodesic generated by $J(0)$, so $J(0)$ is clearly tangent to N. Moreover, the condition $0 = \frac{1}{2}(\|J\|^2)'(0) = \langle J(0), J'(0)\rangle$ implies $J'(0) \in (T_pN)^\perp$. This shows that J is an N-Jacobi field.

To show the last statement, it is enough to prove that if there are no perpendicular Jacobi fields J such that $J(0) = 0$ and there exists $t > 0$ with $(\|J\|^2)'(t) = 0$, then this does not happen for any Jacobi field. The previous hypothesis means that any perpendicular Jacobi field J vanishing at 0 satisfies $(\|J\|^2)'(t) > 0$ for $t > 0$. Now, let $\bar{J}$ be any Jacobi field along a unit speed geodesic c with $\bar{J}(0) = 0$. It decomposes as the sum $\bar{J}(t) = J(t) + \lambda t\dot{c}(t)$ of a perpendicular Jacobi field J and the tangent field $\lambda t\dot{c}(t), \lambda \in \mathbb{R}$. An easy computations yields

$$(\|\bar{J}\|^2)' = (\|J\|^2)' + 2\lambda^2 t,$$

which shows that $(\|\bar{J}\|^2)' > 0$ for positive t, as we wanted to prove. $\square$

1.3 Green subbundles

The purpose of this section is to introduce one of the main tools in the study of the dynamics of geodesic flows in the setting of manifolds without conjugate points, the Green bundles. We first need to understand better the solutions of the Jacobi equation.

Proposition 1.3.1. *Let M be a manifold without conjugate points and let $c : \mathbb{R} \to M$ be a geodesic. Choose $t \neq 0$. For each $X \in T_{c(0)}M$ and $Y \in T_{c(t)}M$ there exists a unique Jacobi field J along c with $J(0) = X$ and $J(t) = Y$. Moreover, J is perpendicular to c if and only if $X \perp \dot{c}(0)$ and $Y \perp \dot{c}(t)$.*

Proof. We consider the liner map from the space of Jacobi fields on γ to $T_{c(0)} \oplus T_{c(t)}$ given by $J \mapsto (J(0), J(t))$. Since M has no conjugate points, nonzero Jacobi fields do not vanish two times, so the kernel of the previous map is trivial. The two spaces have the same dimension, so it is a isomorphism. By formula (1.1) if $J(0)$ and $J(t)$ are orthogonal to the geodesic, then J is orthogonal for all time. □

For $v \in T^1M$ and $X \in T_{\pi(v)}M$ orthogonal to v, we consider for $t \neq 0$ the Jacobi field $J_{v,X,t}$ on the geodesic c_v such that $J_{v,X,t}(0) = X$ and $J_{v,X,t}(t) = 0$.

Proposition 1.3.2. *Let M be a manifold without conjugate points.*

1. *The fields $J_{v,X,t}$ converge uniformly on compact subsets when $t \to +\infty$ ($t \to -\infty$) to an orthogonal Jacobi field $J^s_{v,X}$ ($J^u_{v,X}$) called the stable (unstable) Jacobi field.*

2. *The correspondences $X \to J^s_{v,X}$ and $X \to J^u_{v,X}$ are linear.*

3. *Stable and unstable Jacobi fields on a geodesic do not depend on the initial point of the geodesic,*
$$J^s_{v,X}(\cdot + t) = J^s_{g_t v, J^s_{v,X}(t)}, \quad J^u_{v,X}(\cdot + t) = J^u_{g_t v, J^u_{v,X}(t)}$$
for all $t \in \mathbb{R}$.

4. *Nontrivial stable and unstable Jacobi fields never vanish.*

In order to understand better these Jacobi fields, it is convenient to work in coordinates. We consider an orthonormal parallel vector frame $E_1, \ldots, E_n$, $n = \dim M$, on the geodesic c_v. Moreover, we can assume that $E_1 = \dot{c}_v$. Consider the matrix valued function $\mathbf{R} : \mathbb{R} \to M_{n-1}(\mathbb{R})$ defined for $2 \leq i, j \leq n$ by
$$\mathbf{R} = (\langle R(\dot{c}_v, E_i)\dot{c}_v, E_j\rangle)_{i,j \geq 2}.$$
Given coefficients $\alpha = (\alpha_2, \ldots, \alpha_n) : \mathbb{R} \to \mathbb{R}^{n-1}$, the perpendicular field
$$J = \sum_{i=2}^{n} \alpha_i E_i,$$
is a Jacobi field if and only if α satisfies the equation
$$\alpha'' + \mathbf{R}\alpha = 0.$$
Let us consider the $(n-1)$-matrix equation
$$\mathbf{J}'' + \mathbf{R}\mathbf{J} = 0, \tag{1.6}$$
the columns of $\mathbf{J} : \mathbb{R} \to M_{n-1}(\mathbb{R})$ represent perpendicular Jacobi fields. If for some interval I, $\det \mathbf{J}(t) \neq 0$ for all $t \in I$, then we can consider the matrix $\mathbf{U} : I \to M_{n-1}(\mathbb{R})$ given by $\mathbf{U} = \mathbf{J}'\mathbf{J}^{-1}$, which is a solution of the Ricatti equation
$$\mathbf{U}' + \mathbf{U}^2 + \mathbf{R} = 0 \tag{1.7}$$

on the interval I.

We remark that, given two solutions $\mathbf{A}, \mathbf{B}$ of (1.6), the Wronskian

$$W(\mathbf{A}, \mathbf{B}) = (\mathbf{A}')^T\mathbf{B} - \mathbf{A}^T\mathbf{B}'$$

is constant for all time.

Now we consider, for $t \neq 0$, the solution $\mathbf{B}_t$ of (1.6) with $\mathbf{B}_t(0) = Id$ and $\mathbf{B}_t(t) = 0$. We observe that the Jacobi field $J_{v,X,t}$ is written in coordinates as $\mathbf{B}_t\alpha$, where $X = \sum_{i\geq 2}\alpha_i E_i(0)$. Also, we let $\mathbf{A}$ denote the solution of (1.6) with $\mathbf{A}(0) = 0$ and $\mathbf{A}'(0) = Id$.

Lemma 1.3.3. *For $0 < s < t$, the solution $\mathbf{B}_t$ of the matrix Jacobi equation has the following integral expression:*

$$\mathbf{B}_t(s) = \mathbf{A}(s)\int_s^t \mathbf{A}(u)^{-1}(\mathbf{A}(u)^{-1})^T \mathrm{d}u.$$

Proof. The expression is well defined because $\det \mathbf{A}(u) \neq 0$ for all $u \neq 0$, given that the columns of $\mathbf{A}$ represent linear independent Jacobi fields and $\mathbf{A}(0) = 0$. If we denote the right hand side by $\mathbf{L}_t(s)$, we can compute

$$\mathbf{L}_t' = \mathbf{A}'\mathbf{A}^{-1}\mathbf{L}_t - (\mathbf{A}^{-1})^T,$$

$$\mathbf{L}_t''+\mathbf{R}\mathbf{L}_t = (\mathbf{A}''\mathbf{A}^{-1}+\mathbf{R})\mathbf{L}_t - \mathbf{A}'\mathbf{A}^{-1}\mathbf{A}'\mathbf{A}^{-1}\mathbf{L}_t + \mathbf{A}'\mathbf{A}^{-1}\mathbf{L}_t' + (\mathbf{A}^{-1})^T(\mathbf{A}')^T(\mathbf{A}^{-1})^T = ((\mathbf{A}^{-1})^T(\mathbf{A}')^T - \mathbf{A}'\mathbf{A}^{-1})(\mathbf{A}^{-1})^T.$$

So $\mathbf{L}_t$ is a solution of 1.6 if and only if $\mathbf{A}'\mathbf{A}^{-1}$ is symmetric. The Wronskian

$$W(\mathbf{A}, \mathbf{A}) = (\mathbf{A}')^T\mathbf{A} - \mathbf{A}^T\mathbf{A}'$$

is constant and equal to 0 at $s = 0$, which implies that $\mathbf{A}'\mathbf{A}^{-1}$ is symmetric for all time $s \neq 0$. It is clear that $\mathbf{L}_t(t) = 0$. By writing $\mathbf{A}(s) = s\mathbf{N}(s)$ (so $\mathbf{N}(0) = Id$), we can check that

$$\lim_{s\to 0}\mathbf{L}_t(s) = \lim_{s\to 0} s\mathbf{N}(s)\int_s^t \frac{1}{u^2}\mathbf{N}(u)^{-1}(\mathbf{N}(u)^{-1})^T\mathrm{d}u = \lim_{s\to 0}\mathbf{N}(s)^{-1}(\mathbf{N}(s)^{-1})^T = Id$$

by applying L'Hôpital's rule to the fraction with numerator equal to the integral and denominator equal to $1/s$. Hence $\mathbf{B}_t$ and $\mathbf{L}_t$ coincide on $(0, t)$. □

We can now prove Proposition 1.3.2.

Proof. We only prove the existence and the properties of $J^s_{v,X}$, for $J^u_{v,X}$ it is enough to reverse the time.

1. In view of the remarks above, it is enough to show that the matrices $\mathbf{B}_t$ converge when $t \to \infty$. To show that $\mathbf{B}_t$ converge uniformly on compact subsets it is enough to check that the initial conditions $\mathbf{B}_t(0)$ and $\mathbf{B}_t'(0)$ converge, thanks to the continuity of the solutions of a differential equation with respect to the initial conditions. The quantity $\mathbf{B}_t(0)$ is constant equal to the identity. We observe that the second quantity $\mathbf{B}_t'(0)$ is symmetric because

$$\mathbf{B}_t'(0)^T - \mathbf{B}_t'(0) = W(\mathbf{B}_t, \mathbf{B}_t)(0) = W(\mathbf{B}_t, \mathbf{B}_t)(t) = 0.$$

Moreover,

$$\mathbf{B}_t'(0) - \mathbf{B}_1'(0) = \lim_{s\to 0} \mathbf{B}_t'(s) - \mathbf{B}_1'(s) = \lim_{s\to 0} \mathbf{A}'(s)\mathbf{A}^{-1}(s)(\mathbf{B}_t(s) - \mathbf{B}_1(s)) =$$

$$\lim_{s\to 0} \mathbf{A}'(s)\int_1^t \mathbf{A}(u)^{-1}(\mathbf{A}(u)^{-1})^T \mathrm{d}u = \int_1^t \mathbf{A}(u)^{-1}(\mathbf{A}(u)^{-1})^T \mathrm{d}u.$$

The integrand $\mathbf{A}(u)^{-1}(\mathbf{A}(u)^{-1})^T$ is symmetric. It is also positive definite for small $u > 0$ (because $\mathbf{A}(u) = u\mathbf{N}(u)$ with $\mathbf{N}(0) = Id$) and hence for all $u > 0$ given that $\det(\mathbf{A}(u)^{-1}(\mathbf{A}(u)^{-1})^T) \neq 0$. This implies that $\mathbf{B}_t'(0)$ is strictly increasing in t.

We will prove that $\mathbf{B}_{-1}'(0) - \mathbf{B}_t'(0)$ is symmetric and positive definite, so $\mathbf{B}_t'(0)$ is bounded and it needs to converge.

We remark that

$$\mathbf{B}_{-1}(s) = -\mathbf{A}(s)\mathbf{A}^{-1}(-1)\mathbf{B}_t(-1) + \mathbf{B}_t(s)$$

so $\mathbf{B}_{-1}'(0) - \mathbf{B}_t'(0) = -\mathbf{A}^{-1}(-1)\mathbf{B}_t(-1)$. Moreover,

$$(\mathbf{B}_t^{-1}\mathbf{A})' = \mathbf{B}_t^{-1}(\mathbf{B}_t^{-1})^T W(\mathbf{B}_t, \mathbf{B}_t)\mathbf{B}_t^{-1}\mathbf{A} - \mathbf{B}_t^{-1}(\mathbf{B}_t^{-1})^T W(\mathbf{B}_t, \mathbf{A}) = \mathbf{B}_t^{-1}(\mathbf{B}_t^{-1})^T,$$

since $W(\mathbf{B}_t, \mathbf{B}_t)(t) = 0$ and $W(\mathbf{B}_t, \mathbf{A})(0) = -Id$. We can now write

$$-(\mathbf{B}_t^{-1}\mathbf{A})(-1) = (\mathbf{B}_t^{-1}\mathbf{A})(0) - (\mathbf{B}_t^{-1}\mathbf{A})(-1) = \int_{-1}^0 \mathbf{B}_t^{-1}(\mathbf{B}_t^{-1})^T \mathrm{d}s,$$

which shows that $-(\mathbf{B}_t^{-1}\mathbf{A})(-1)$ is symmetric and positive definite, since the integrand is equal to the identity at 0 and $\det(\mathbf{B}_t^{-1}(\mathbf{B}_t^{-1})^T) \neq 0$. Its inverse is $\mathbf{B}_{-1}'(0) - \mathbf{B}_t'(0)$, which also needs to be symmetric and positive definite. So $\mathbf{B}_t$ converges to a solution of (1.6) which we denote by $\mathbf{B}^s$.

2. In coordinates, the stable orthogonal Jacobi field with initial condition α can be written as $\mathbf{B}^s(s)\alpha$, hence it is linear in α.

3. In coordinates,

$$u \mapsto \mathbf{B}_t(u)\mathbf{B}_t(s)^{-1}\mathbf{B}^s(s)\alpha$$

is the Jacobi field which is equal to $\mathbf{B}^s(t)\alpha$ at $u = s$ and 0 at $u = t$, so when t tends to $+\infty$ it converges to the field $J^s_{g_s v, J^s_{v,X}(s)}(u - s)$. On the other hand, we see that it converges to $u \mapsto \mathbf{B}^s(u)\alpha$. This proves the property.

4. If for some s we have $J^s_{v,X}(s) = 0$, by the previous property

$$J^s_{v,X}(\cdot + s) = J^s_{g_s v,0} = 0,$$

so $X = 0$.

□

If we add stronger assumptions on the manifold M, we can say more about the behavior of stable and unstable Jacobi fields. First of all, we look at curvature restrictions on M. Given a geodesic c, we refer to the sectional curvatures of planes passing through a point $c(t)$ and containing the tangent vector $\dot{c}(t)$ as the curvatures along c.

Lemma 1.3.4. *If the curvatures along a geodesic c are bounded above by a constant $-a^2 < 0$, then every orthogonal Jacobi field J with $J(0) = 0$ satisfies*

$$||J||'(s) \geq a \coth{(as)}||J(s)||$$

for $s > 0$. In particular, for $t > s > 0$,

$$\frac{||J(t)||}{||J(s)||} \geq \frac{\sinh(at)}{\sinh(as)}.$$

Proof. By Lemma 1.2.9, the quantity

$$\big(\|J\|'(t)\sinh(at) - a\,\|J\|(t)\cosh(at)\big)' = \big(\|J\|''(t) - a^2\,\|J\|(t)\big)\sinh(at).$$

is nonnegative for positive time t. Then, for $t > 0$,

$$\|J\|'(t)\sinh(at) - a\,\|J\|(t)\cosh(at) \geq 0, \tag{1.8}$$

and the first statement follows. Integration of $||J||'/||J||$ between s and t yields the second formula. ☐

Lemma 1.3.5. *If the curvatures along a geodesic c are bounded below by a constant $-b^2 < 0$, then every orthogonal Jacobi field J with $J(0) = 0$ satisfies*

$$||J'||(s) \leq b \coth{(bs)}||J(s)||$$

for $s > 0$. In particular, for $t > s > 0$,

$$\frac{||J(t)||}{||J(s)||} \leq \frac{\sinh(bt)}{\sinh(bs)}.$$

Proof. Recall that $J(t)$, in coordinates, is written as $\mathbf{A}(t)\alpha$ for some $\alpha \in \mathbb{R}^{n-1}$ and its derivative $J'(t)$ is identified with $\mathbf{A}'(t)\alpha$. Then $\mathbf{U}(t) = \mathbf{A}'(t)\mathbf{A}(t)^{-1}$ is a symmetric solution of the Ricatti equation (1.7) defined for $t > 0$ as we saw in the proof of Proposition 1.3.2. So it will be enough to prove that $||\mathbf{U}(t)|| \leq b\coth(bt)$.

Fix $x \in \mathbb{R}^{n-1}$ with $||x|| = 1$. We will prove that for all $t > 0$,

1. $\langle \mathbf{U}(t)x, x\rangle \leq b\coth(bt))$,
2. $\langle \mathbf{U}(t)x, x\rangle \geq -b$,

which immediately imply the desired result.

1. In fact, we will prove that $\langle \mathbf{U}(t)x, x\rangle \leq \kappa\coth(\kappa(t-\varepsilon))$ for all $\kappa > b$, $\varepsilon > 0$ and $t > \varepsilon$, and then pass to the limit.

We consider the function

$$f(t) = \langle(\mathbf{U}(t) - \mathbf{V}(t))x, x\rangle = \langle \mathbf{U}(t)x, x\rangle - \kappa\coth(\kappa(t-\varepsilon)),$$

where $\mathbf{V}(t) = \kappa\coth(\kappa(t-\varepsilon))Id$ is a solution of the Ricatti equation

$$\mathbf{V}' + \mathbf{V}^2 - \kappa^2 Id = 0$$

defined for $t \neq \varepsilon$. We claim that if for some $t_0 \neq \varepsilon$, $f(t_0) = 0$ then $f'(t_0) < 0$. This means that if f gets strictly negative on (ε, ∞) it stays strictly negative. The derivative is

$$f'(t) = \langle(\mathbf{U}'(t) - \mathbf{V}'(t))x, x\rangle = -\langle\mathbf{U}^2(t)x, x\rangle + \langle\mathbf{V}^2(t)x, x\rangle - (\langle\mathbf{R}(t)x, x\rangle + b^2) <$$
$$-||\mathbf{U}(t)x||^2 + ||\mathbf{V}(t)x||^2,$$

since $\langle\mathbf{R}(t)x, x\rangle$ is the curvature on a plane tangent to $\dot{c}(t)$, which is greater than $-b^2 > -\kappa^2$. If $f(t_0) = 0$, we have

$$||\mathbf{V}(t)x||^2 = (\kappa\coth(\kappa(t-\varepsilon)))^2 = (\langle\mathbf{U}(t)x, x\rangle)^2 \leq ||\mathbf{U}(t)x||^2,$$

so $f'(t_0) < 0$. But now observe that $\lim_{t\to\varepsilon^+} f(t) = -\infty$. We conclude that f is strictly negative for all $t > \varepsilon$, or equivalently,

$$\langle\mathbf{U}(t)x, x\rangle < \kappa\coth(\kappa(t-\varepsilon)).$$

2. We proceed by contradiction. Assume that $\langle\mathbf{U}(t_1)x, x\rangle < -b$ for some $t_1 > 0$ and take $-\kappa$ between both values. Then for very large $\varepsilon > t_1$,

$$f(t_1) = \langle\mathbf{U}(t_1)x, x\rangle - \kappa\coth(\kappa(t_1-\varepsilon)) < 0.$$

The same claim made before about f now implies $f(t) < 0$ for all $t_1 \leq t < \varepsilon$. But taking the limit of $f(t)$ when $t \to \varepsilon^-$ gives $\langle\mathbf{U}(\varepsilon)x, x\rangle + \infty$, which should be nonpositive, thus producing a contradiction.

In order to obtain the second equality, we remark that $|||J||'| \leq ||J'||$ and proceed as in Lemma 1.3.4. □

The previous estimates allow us to obtain information about stable and unstable Jacobi fields under curvature constraints.

Theorem 1.3.6. *Let c be a geodesic of M.*

1. *If the curvatures along c are bounded above by a constant $-a^2 \leq 0$, then for every stable Jacobi J field along c, we have for all $t \geq 0$*

$$||J(t)|| \leq ||J(0)||e^{-at}, \quad ||J'(t)|| \geq a||J(t)||.$$

2. *If the curvatures along c are bounded below by a constant $-b^2 \leq 0$, then for every stable Jacobi J field along c, we have for all $t \geq 0$*

$$||J(t)|| \geq ||J(0)||e^{-bt}, \quad ||J'(t)|| \leq b||J(t)||.$$

If J is unstable, we will have the corresponding estimates for negative time.

Proof. 1. We consider the Jacobi field $K_n(t) = J_{v,J(0),n}(n-t)$ on the geodesic $t \mapsto c(n-t)$. Clearly K_n is an orthogonal Jacobi field with $K_n(0) = 0$.

A direct application of Lemma 1.3.4 to K_n (replacing s for $n-t$ and t for n) gives

$$\frac{||J(0)||}{||J_{v,J(0),n}(t)||} \geq \frac{\sinh(an)}{\sinh(a(n-t))}, \quad ||J_{v,J(0),n}||'(t) \geq a\coth(a(n-t))||J_{v,J(0),n}(t)||$$

Observe that $||J||'||J|| = \langle J, J'\rangle$, so $|||J||'| \leq ||J'||$ whenever $||J|| \neq 0$ for any field J. Taking the limit when $n \to +\infty$ gives the desired result.

2. We argue similarly using Lemma 1.3.5. □

Next, we want to say something about the growth of the differential of the geodesic flow through these stable and unstable Jacobi fields. The subspaces of the tangent space of T^1M corresponding to these stable and unstable Jacobi fields are called Green spaces.

Definition 1.3.1. We define the stable Green bundle G^s as the subbundle of TT^1M over T^1M whose fibers in the double tangent bundle decomposition are given by

$$G^s(v) = \{(J^s_{v,X}(0), {J^s_{v,X}}'(0)) \,|\, X \in v^\perp\} \subset T_vT^1M.$$

Similarly, we define the unstable Green bundle G^u by

$$G^u(v) = \{(J^u_{v,X}(0), {J^u_{v,X}}'(0)) \,|\, X \in v^\perp\} \subset T_vT^1M.$$

We remark that the fibers of G^s and G^u have each dimension $n-1$, that they are perpendicular to the direction of the geodesic flow, and that this flow leaves the subbundles invariant. However, in the general case $G^s(v)$ and $G^u(v)$ are not necessarily linearly independent.

We recall the definition of the Anosov property for a geodesic flow and the Sasaki metric.

Definition 1.3.2. Let M be a complete Riemannian manifold. We say that the geodesic flow g_t on T^1M is Anosov if there exists a proper g_t-invariant splitting

$$TT^1M = E^s \oplus E^u \oplus \mathbb{R}\, G,$$

where G is the direction of the geodesic flow, and there exist constants $C > 0$ and $\alpha > 0$ such that for all $t \geq 0$ we have, in the Sasaki metric,

$$||d_v g_t(Z)|| \leq C e^{-\alpha t} ||Z||, \quad \forall Z \in E^s(v),$$

$$||d_v g_{-t}(Z)|| \leq C e^{-\alpha t} ||Z||, \quad \forall Z \in E^u(v).$$

Theorem 1.3.7. *Let M be a manifold with curvature bounded between two negative constants $-b^2 \leq K \leq -a^2 < 0$. Then the geodesic flow of M is Anosov.*

Proof. The splitting is given by the Green subbundles. Thanks to Theorem 1.3.6 and equation (1.2) we can control the growth of the differential of the geodesic flow:

$$||d_v g_\pm(Z)|| \leq \sqrt{1+b^2}\, e^{-at} ||Z||$$

for $Z \in G^{s/u}(v)$. This and the g_t-invariance imply that G^s and G^u are linearly independent, so they are complementary because of their dimensions. □

When the sectional curvature are not negatively pinched, there are examples of both Anosov and non Anosov geodesic flows. The most general characterization of Anosov geodesic flows on compact manifolds without conjugate points was given by Eberlein.

Theorem 1.3.8. *[Ebe73b] Let M be a compact Riemannian manifold without conjugate points. The following conditions are equivalent:*

1. *The geodesic flow g_t of M is Anosov.*
2. *For every $v \in T^1M$, $G^s(v) \cap G^u(v) = \{0\}$.*

3. There are no nontrivial perpendicular Jacobi fields bounded for all time.

Moreover, if g_t is Anosov, the spaces $E^s(v)$ and $E^u(v)$ of the splitting in Definition 1.3.2 coincide with $G^s(v)$ and $G^u(v)$, respectively.

We end the section with some remarks about the behaviour and the regularity of stable and unstable Jacobi fields. Eberlein proved that Jacobi fields bounded in positive time are stable.

Proposition 1.3.9. *[Ebe73b, Proposition 2.12] Let M be a manifold without conjugate points and curvature bounded below by a constant $-b^2 \leq 0$. If J is a perpendicular Jacobi field with $||J(t)||$ bounded for $t \geq 0$, then J is stable.*

The converse of Proposition 1.3.9 is not true in general. The next result gives a condition on the growth of vanishing Jacobi fields under which stable Jacobi fields are bounded. Nonpositively curved manifolds and manifolds without focal points satisfy this condition.

Proposition 1.3.10. *[Ebe73b, Proposition 2.13] Let $\tilde{M}$ be a manifold without conjugate points and curvature bounded below by a constant $-b^2 \leq 0$. Assume that there are constants $C, T > 0$ such that every Jacobi field on a unit speed geodesic of $\tilde{M}$ with $J(0) = 0$ and every $t > s \geq T$ satisfy*

$$||J(t)|| \geq C||J(s)||.$$

Then the following are true:

1. *A perpendicular Jacobi field J is stable if and only if $||J(t)||$ is bounded for $t \geq 0$.*

2. *The Green bundles are continuous.*

Proposition 1.3.10 also asserts the continuity of the Green bundles. These subbundles are just measurable in general. There is an example of a compact surface without conjugate points but with discontinuous Green subbundles [BBB87]. In Chapter 6, we will assume that they are continuous.

Chapter 2

Structure of manifolds without conjugate points

In this chapter, we continue with the presentation of different tools and concepts that we will use in the second part of the thesis. These objects rely on the global geometry of manifolds without conjugate points. The so-called boundary at infinity puts some light on the structure of the set of geodesics in the universal cover of a manifolds without conjugate points.

We will define also two central objects in our works, the stable and the unstable horospheres. These horospheres form two foliations invariant by the geodesic flow. Moreover, they are the natural candidates for the stable and the unstable manifolds of this flow, although they are not equal in general. The main results of this thesis (Chapters 4 to 6) are about the equidistribution under the action of the geodesic flow and the unique ergodicity of horospherical foliations.

In Section 2.1, we introduce two hypothesis, the quasi-convexity and the divergence of geodesic rays, satisfied by a large class of manifolds without conjugate points. We define the boundary at infinity in Section 2.2, and prove a structure theorem under the previous hypothesis. Next, in Section 2.3, we discuss visibility manifolds, which we will need later in the text. The approach used to define horospheres are Busemann functions, as explained in Section 2.4. In Section 2.5, we address the existence of strips of geodesics, that is, sets of geodesics which are at distance bounded for all time. In Section 2.6, we introduce the rank 1 condition, which will be used in the sequel, mainly in the setting of nonpositive curvature. Finally, in Section 2.7, we display a diagram with the main classes of manifolds discussed in this chapter and the relations between them.

2.1 Quasi-convexity and divergence of geodesic rays

We will add hypothesis on our manifold so that later we can define a boundary at infinity with interesting properties. We start with a very general setting, and later we will add hypothesis in order to strengthen the results. Proceeding in this way we will treat simultaneously the different situations that we will find in Part II. Let us begin by defining the classes of manifolds on which we will work.

Definition 2.1.1. Let M be a manifold without conjugate points. We say that $\tilde{M}$ is quasi-convex if there exist constants $A, B > 0$ such that for every two geodesic segments $c_i : [a_i, b_i] \to \tilde{M}$, $i = 1, 2$,

$$d_H(c_1([a_1, b_1]), c_2([a_2, b_2])) \leq A \max\{d(c_1(a_1), c_2(a_2)), d(c_1(b_1), c_2(b_2))\} + B,$$

where d_H is the Hausdorff distance.

Two geodesic rays $c_1, c_2 : \mathbb{R}_+ \to \tilde{M}$ with $c_1(0) = c_2(0)$ diverge if

$$\lim_{t \to +\infty} d(c_1(t), c_2(t)) = +\infty.$$

We say that geodesic rays on $\tilde{M}$ diverge uniformly if for every $R > 0$ and $\theta > 0$, there exists $T > 0$ such that for any two geodesic rays $c_1, c_2 : \mathbb{R} \to \tilde{M}$ with $c_1(0) = c_2(0)$ and $\angle(\dot{c}_1(0), \dot{c}_2(0)) \geq \theta$, we have

$$\forall t \geq T, \quad d(c_1(t), c_2(t)) \geq R.$$

The combination of these two properties alone has strong consequences on the structure of the manifold. It is not difficult to prove the following lemma concerning manifolds with a compact quotient.

Lemma 2.1.1. *Let M be a compact manifold without conjugate points. If the universal cover $\tilde{M}$ is quasi-convex and geodesic rays on $\tilde{M}$ diverge, then the divergence is uniform.*

The first important family of examples of quasi-convex manifolds is given in the next proposition. The condition on the growth of vanishing Jacobi fields already appeared in Proposition 1.3.10.

Proposition 2.1.2. *Let $\tilde{M}$ be a manifold without conjugate points. Assume that there are constants $C, T > 0$ such that every Jacobi field on a unit speed geodesic of $\tilde{M}$ with $J(0) = 0$ and every $t > s \geq T$ satisfy*

$$||J(t)|| \geq C||J(s)||.$$

Then $\tilde{M}$ is quasi-convex. In particular, if $\tilde{M}$ has no focal points, it is quasi-convex.

Proof. First, we observe that it is enough to check that there exist constants $A, B \geq 0$ such that any two unit speed geodesic segments $c_i : [0, l_i] \to \tilde{M}$, $i = 1, 2$ with $c_1(0) = c_2(0)$ we have

$$d_H(c_1([a_1, b_1]), c_2([a_2, b_2])) \leq Ad(c_1(l_1), c_2(l_2)) + B.$$

Actually, if we have any two geodesics $c_i : [a_i, b_i] \to \tilde{M}$, $i = 1, 2$, we can apply the previous inequality to the couples (c_1, c) and (c, c_2), where c is the geodesic joining $c_1(a_1)$ to $c_2(b_2)$ and obtain

$$d_H(c_1, c_2) \leq d_H(c_1, c) + d_H(c, c_2) \leq A(d(c_1(b_1), c_2(b_2)) + d(c_1(a_1), c_2(a_2))) + 2B.$$

So $\tilde{M}$ is $(2A, 2B)$-quasi-convex.

We consider the geodesic $\alpha : [0, 1] \to \tilde{M}$ joining $c_1(l_1)$ to $c_2(l_2)$ and denote its length by $d = d(c_1(l_1), c_2(l_2))$. Set $p = c_1(0) = c_2(0)$. In terms of the exponential map, α is written as $\alpha(s) = \exp_p(V(s))$ for some vector field $V : [0, 1] \to T_p\tilde{M}$. Also consider the parametrized surface $f(t, s) = \exp_p(tV(s))$, $(t, s) \in [0, 1]^2$, (see Figure 2.1) and the family of Jacobi fields $J_s = \frac{\partial f}{\partial s}$ on the family of non normalized geodesics $t \mapsto f(t, s)$.

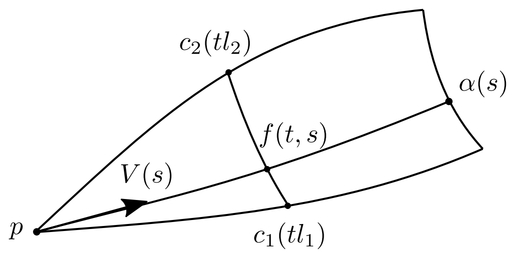


Figure 2.1: Parametrized surface f.

For $0 \le t \le 1$,

$$d(c_1(tl_1), c_2(tl_2)) \le l(s \mapsto f(t,s)) = \int_0^1 ||J_s(t)|| \mathrm{d}s.$$

The hypothesis tells us after normalizing that $||J_s(t)|| \le ||J_s(1)||/C$ if $d(p, f(t,s)) = td(p, \alpha(s)) \ge T$.

Now, if $tl_1 \ge T + d$, then for all t,

$$d(p, f(t,s)) = td(p, \alpha(s)) \ge t(l_1 - d(c_1(l_1), \alpha(s))) \ge t(l_1 - d) \ge T,$$

which implies

$$d(c_1(tl_1), c_2(tl_2)) \le \frac{1}{C} \int_0^1 ||J_s(1)|| \mathrm{d}s = \frac{d}{C}.$$

Otherwise, $tl_1 \le T + d$ and

$$\begin{aligned} d(c_1(tl_1), c_2(tl_2)) \le d(c_1(tl_1), p) + d(p, c_2(tl_2)) = tl_1 + tl_2 \le \\ tl_1 + t(l_1 + d) \le 2(T + d) + td \le 3d + 2T. \end{aligned}$$

So we have proved the claim with $A = \max(3, 1/C)$ and $B = 2T$.

Finally, we observe that the absence of focal points implies the condition in the statement with $C = 1$ and $T = 0$, because the norm of vanishing Jacobi fields is non-decreasing for positive times. □

It is also true that the condition in the previous proposition, plus a lower curvature bound imply that geodesic rays diverge uniformly. In fact, Eschenburg and O'Sullivan proved the uniform divergence of geodesic rays for manifolds with continuous Green bundles and curvature bounded below.

Proposition 2.1.3. *[EO76, Corollary 1] Let $\tilde{M}$ be a simply connected manifold without conjugate points and the curvature bounded below by a constant $-b^2$. Assume that the Green bundles are continuous. Then geodesic rays diverge uniformly.*

It is well known that the condition of Proposition 2.1.2 implies the continuity of Green bundles (see bounded asymptote condition in [Kni86, Chapter V]).

The second important family of examples of quasi-convex spaces is given by manifolds whose universal cover is Gromov hyperbolic.

Definition 2.1.2. Three geodesic segments $c_i : [a_i, b_i] \to \tilde{M}, i = 1, 2, 3$ form a geodesic triangle

$$\Delta = c_1([a_1, b_1]) \cup c_2([a_2, b_2]) \cup c_3([a_3, b_3])$$

if $c_1(b_1) = c_2(a_1)$, $c_2(b_2) = c_3(a_3)$ and $c_3(b_3) = c_1(a_1)$. Given $\delta > 0$, the triangle Δ is called δ-thin if for every $i \in \{1, 2, 3\}$, and every $t \in [a_i, b_i]$, the point $c_i(t)$ is in the δ-neighborhood of

$$\cup_{k \neq i} c_k([a_k, b_k]).$$

The space $\tilde{M}$ is δ-hyperbolic if all the triangles are δ-thin.

Proposition 2.1.4. *If $\tilde{M}$ is δ-hyperbolic, then $\tilde{M}$ is $(1, 2\delta)$-quasi-convex.*

Proof. Let $c_i : [a_i, b_i] \to \tilde{M}$, $i = 1, 2$ be two geodesics. Denote the geodesic joining $c_1(a_1)$ to $c_2(a_2)$ by α, the geodesic joining $c_1(b_1)$ to $c_2(b_2)$ by β and the geodesic joining $c_1(a_1)$ to $c_2(b_2)$ by γ. Consider the triangles $c_1\beta\gamma$ and $c_2\gamma\alpha$ (Figure 2.2).

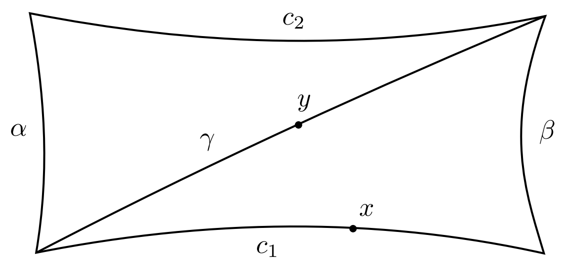


Figure 2.2: Triangles $c_1\beta\gamma$ and $c_2\gamma\alpha$.

Given $x \in c_1$, by δ-hyperbolicity, either $d(x, \beta) \leq \delta$ or $d(x, \gamma) \leq \delta$. In the first case, we will have

$$d(x, c_2) \leq d(x, c_2(b_2)) \leq d(x, \beta) + d(c_1(b_1), c_2(b_2)) \leq d(c_1(b_1), c_2(b_2)) + \delta.$$

In the second, let $y \in \gamma$ such that $d(x, \gamma) = d(x, y)$. Again by hyperbolicity, either $d(y, \alpha) \leq \delta$ or $d(y, c_2) \leq \delta$. So either,

$$d(x, c_2) \leq d(x, c_2(a_2)) \leq d(x, y) + d(y, \alpha) + d(c_1(a_1), c_2(a_2)) \leq d(c_1(a_1), c_2(a_2)) + 2\delta$$

or

$$d(x, c_2) \leq d(x, y) + d(y, c_2) \leq 2\delta.$$

Hence,

$$d(x, c_2) \leq \max\{d(c_1(a_1), c_2(a_2)), d(c_1(b_1), c_2(b_2))\} + 2\delta.$$

Similarly, we can bound $d(c_1, z)$ for all $z \in c_2$. We conclude that $\tilde{M}$ is quasi-convex with the right constants. □

2.2 Boundary at infinity

The first important tool that we present is the boundary at infinity. Inspired by hyperbolic geometry, this boundary was studied by M. Gromov for Gromov hyperbolic spaces [Gro87, Gro81], and by P. Eberlein for visibility manifolds [Ebe72]. For nonpositively curved spaces it has also been studied, for example by W. Ballmann, M. Gromov and V. Schroeder in [BGS85].

Definition 2.2.1. Let $\tilde{M}$ be a simply connected, complete manifold without conjugate points. A geodesic defined on $\mathbb{R}_+$ is called a geodesic ray. Two geodesic rays $\sigma_1, \sigma_2 : \mathbb{R}_+ \to \tilde{M}$ are said to be *asymptotic* if there exists $C > 0$ such that for all $t \geq 0$, we have $d(\sigma_1(t), \sigma_2(t)) \leq C$. Since being asymptotic is an equivalence relation, we can consider the set $\partial\tilde{M}$ of equivalence classes of geodesic rays on $\tilde{M}$. This set is called the *boundary at infinity*.

To a vector $v \in T^1\tilde{M}$, we can associate two points v_+ and v_- at infinity, which are respectively the classes in $\partial\tilde{M}$ of the positive and the negative ray generated by v. Thanks to our assumption that $\tilde{M}$ has no conjugate points, any two points $x, y \in \tilde{M}$, $x \neq y$ can be joined by a unique geodesic. We denote by $V(x, y)$ the unique vector in $T^1_x\tilde{M}$ tangent to the geodesic joining x to y. It is natural then to ask whether we can join points at the boundary. In negative curvature, it is well-known that any point at infinity can also be joined to both a point in the interior and to another point at infinity by a unique geodesic. As we will see, the situation is somewhat different outside the negative curvature case, and everything is not completely understood yet.

Theorem 2.2.1. *Let $\tilde{M}$ be a simply connected, complete manifold without conjugate points.*

1. *If $\tilde{M}$ is quasi-convex, then for every $x \in \tilde{M}$ and $\xi \in \partial\tilde{M}$, there exists at least a vector $v \in T^1_x\tilde{M}$ such that $v_+ = \xi$.*
2. *If geodesic rays on $\tilde{M}$ diverge, then for every $x \in \tilde{M}$ and $\xi \in \partial\tilde{M}$, there is at most a vector $v \in T^1_x\tilde{M}$ such that $v_+ = \xi$.*

Assume that $\tilde{M}$ is quasi-convex and that geodesic rays diverge. We denote by $V(x, \xi)$ the unique vector in $T^1_x\tilde{M}$ pointing to ξ.

3. *There is a topology on $\bar{M} := \tilde{M} \cup \partial\tilde{M}$ which extends that of $\tilde{M}$, with a basis formed by open sets of $\tilde{M}$ together with sets of the form*
$$T_{v,\varepsilon,R} := \{q \in \bar{M} \mid \angle(V(\pi(v), q), v) < \varepsilon \text{ and } d(\pi(v), q) > R \text{ if } q \in \tilde{M}\}.$$
with $v \in T^1\tilde{M}$, $\varepsilon, R > 0$.
4. *The map $\{(x, y) \in \tilde{M} \times \bar{M} \mid x \neq y\} \to T^1\tilde{M}$, $(x, \xi) \mapsto V(x, \xi)$ is continuous. Its restriction to $\tilde{M} \times \partial\tilde{M} \to T^1\tilde{M}$ is a homeomorphism. Moreover, $\bar{M}$ is topologically a closed disk, $\partial\tilde{M}$ corresponds to the boundary and $\tilde{M}$ to the interior of the disk.*

Proof. 1. Consider a geodesic c in the class ξ. For $t > 0$, let $v_t \in T^1_x\tilde{M}$ be the vector tangent to the geodesic joining x to $c(t)$. By quasi-convexity, for $t > 0$ we have
$$d_H(c([0, t], c_{v_t}([0, d(x, c(t))])) \leq Ad(c(0), x) + B.$$
Let v be an accumulation point of the sequence v_t when $t \to +\infty$. Passing the inequality to the limit we obtain that c and c_v are at bounded Hausdorff distance.

Claim: If the Hausdorff distance between two geodesic rays c, c' is finite, then $d(c(t), c'(t))$ is bounded.

Proof. There exists a constant $K > 0$ such that for every $t > 0$, there exists $s_t > 0$ with $d(c(t), c'(s_t)) \leq K$. Consider the quadrilateral $c(0)c'(0)c'(s_t)c(t)$. The lengths of its sides are $d(c(0), c'(0))$, s_t, $d(c'(s_t), c(t))$ and t and we have the inequality
$$|s_t - t| \leq d(c(0), c'(0)) + d(c'(s_t), c(t)) \leq d(c(0), c'(0)) + K,$$
therefore
$$\begin{aligned} d(c(t), c'(t)) &\leq d(c(t), c'(s_t)) + d(c'(s_t), c'(t)) \\ &= d(c(t), c'(s_t)) + |s_t - t| \leq d(c(0), c'(0)) + 2K. \end{aligned}$$
□

2. If geodesic rays diverge, the distance between the geodesic generated by two distinct vectors $v, w \in T^1_x\tilde{M}$ goes to infinity, so they are not asymptotic and $v_+ \neq w_+$.

3. Clearly the family $\mathcal{B}$ formed by open subsets of $\tilde{M}$ together with the truncated cones $T_{v,\varepsilon,R}$ is a cover of $\bar{M}$. To see that $\mathcal{B}$ is the basis of some topology, we need to see that for any $x \in U \cap V$, where $U, V \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U \cap V$.

The following observations prove the previous property:

- For every truncated cone $T_{v,\varepsilon,R}$ and every $y \in T_{v,\varepsilon,R}$, we have for small enough $\varepsilon' > 0$ (depending on v, ε and y), $T_{V(\pi(v),y),\varepsilon',R} \subset T_{v,\varepsilon,R}$.

- If $y \in T_{v,\varepsilon,R} \cap \tilde{M}$, there exists $\varepsilon' > 0$ such that $B(y, \varepsilon') \subset T_{v,\varepsilon,R}$.

  Proof. Thanks to the previous property we can assume that y is in the positive geodesic ray generated by v, so $y = c_v(t_0), t_0 > R$. If $\varepsilon' < t_0 - R$, the ball $B(y, \varepsilon')$ is at distance at least R from $\pi(v)$. Since the map $z \mapsto V(\pi(v), z)$ is continuous, if $z \in B(y, \varepsilon')$ and $\varepsilon' > 0$ is small enough, then the angle between v and $V(\pi(v), z)$ is less than ε. So $B(y, \varepsilon') \subset T_{v,\varepsilon,R}$. □

- If $\xi \in T_{v,\varepsilon,R} \cap \partial\tilde{M}$, for any $y \in \tilde{M}$, there exists $\varepsilon' > 0$ and $R' > 0$ such that $T_{V(y,\xi),\varepsilon',R'} \subset T_{v,\varepsilon,R}$.

  Proof. We can assume that $v_+ = \xi$. We proceed by contradiction. If the property is false, there is a sequence q_n of points of $\bar{M}$ with $d(y, q_n) \to +\infty$ (understanding that $d(y, q_n) = +\infty$ if $q_n \in \partial\tilde{M}$) such that $\angle(V(y,\xi), V(y, q_n))$ $\to 0$ but
  $$\angle(v, V(\pi(v), q_n)) \geq \varepsilon. \tag{2.1}$$
  Now, if $q_n \in \tilde{M}$, let α_n be the geodesic segment between $\pi(v)$ and q_n and β_n be the one between y and q_n. By quasi-convexity we have
  $$d_H(\alpha_n, \beta_n) \leq Ad(\pi(v), y) + B. \tag{2.2}$$
  The same is true if $q_n \in \partial\tilde{M}$ and α_n, β_n are replaced respectively by the geodesic rays from $\pi(v), y$ and in the asymptotic class q_n, as we saw in the proof of 1.

  Let w be an accumulation point of the $V(\pi(v), q_n)$. Passing to the limit in equation (2.2), we obtain that the geodesic rays generated by w and $V(y,\xi) = \lim V(y, q_n)$ are at bounded Hausdorff distance. Therefore they must be asymptotic. By the uniqueness of the ray from $\pi(v)$ in the class ξ we deduce $v = w$. But (2.1) gives in the limit $\angle(v, w) \geq \varepsilon > 0$, so we get a contradiction. □

4. It is clear that the map is continuous at points $(x, y) \in \tilde{M} \times \tilde{M}, x \neq y$. Let $(x, \xi) \in \tilde{M} \times \partial\tilde{M}$ and (x_n, ξ_n) a sequence of $\{(x, y) \in \tilde{M} \times \bar{M} \mid x \neq y\}$ converging to (x, ξ). By definition of the topology of $\bar{M}$, we know that the angle between $V(x, \xi)$ and $V(x, \xi_n)$ tends to 0, so $V(x, \xi_n)$ converges to $V(x, \xi)$. Moreover, the Hausdorff distance between the geodesic segments or rays joining x to ξ_n and x_n to ξ_n is less than $Ad(x, x_n) + B$. So any accumulation point of $V(x_n, \xi_n)$ generates

a geodesic from x asymptotic to the geodesic generated by $V(x,\xi)$. So $V(x_n,\xi_n)$ converges to $V(x,\xi)$.

It is clear that $V|_{\tilde{M}\times\partial\tilde{M}}$ is bijective. Its inverse is the map $v\mapsto(\pi(v),v_+)$. The first component is clearly continuous. Let v_n be a sequence of vectors converging to v in $T^1\tilde{M}$. We want to see that $v_{n+}\to v_+$. Consider the vector $V(\pi(v),v_{n+})$, whose geodesic is asymptotic to the geodesic generated by v_n, and the Hausdorff distance is controlled by $Ad(\pi(v),\pi(v_n))+B$. Since the geodesic generated by any accumulation point of the $V(\pi(v),v_{n+})$ must be asymptotic to the geodesic generated by v, we see that in fact $V(\pi(v),v_{n+})$ has to converge to v. So the angle between $V(\pi(v),v_{n+})$ and v goes to 0. This shows that v_{n+} converges to v_+.

Let φ be any increasing homeomorphism between $[0,+\infty]$ and $[0,1]$. For any $p\in\tilde{M}$, we define the map f_p between $\bar{M}$ and the closed unit ball $B^1_p\tilde{M}$ of $T_p\tilde{M}$ by

$$f_p(x)=\varphi(d(p,x))V(p,x),$$

understanding that $f_p(p)=0$. The previous steps show that this map is continuous. Its inverse is given by $f_p^{-1}(0)=p$, $f_p^{-1}(v)=\exp_p(\varphi^{-1}(||v||)v/||v||)$ if $0<||v||<1$ and $f_p^{-1}(v)=v_+$ if $||v||=1$. The map is clearly continuous on the open unit ball. Moreover, if $v_n\to v$ with $||v||=1$, we have $d(p,f_p^{-1}(v_n))\to+\infty$ and $\angle_p(f_p^{-1}(v),f_p^{-1}(v_n))=\angle(v_n,v)\to 0$ so $f_p^{-1}(v_n)\to f_p^{-1}(v)$ and we have shown that the inverse is continuous. Therefore f_p is a homeomorphism. □

2.3 Visibility manifolds

The next class of manifolds that we will talk about was introduced by P. Eberlein and B. O'Neill in [EO73] for nonpositively curved manifolds and by P. Eberlein alone in [Ebe72] for manifolds without conjugate points in general.

Definition 2.3.1. We say that a simply connected manifold $\tilde{M}$ without conjugate points satisfies the visibility axiom if for every $\varepsilon>0$, there exists $R>0$ such that for every geodesic segment $\sigma:[a,b]\to\tilde{M}$ and every $p\in\tilde{M}$,

$$d(\sigma([a,b]),p)\geq R\implies\angle_p(\sigma(a),\sigma(b))\leq\varepsilon.$$

We say that a manifold M without conjugate points is a visibility manifold if the universal cover $\tilde{M}$ satisfies the visibility axiom above.

Some authors call this property the uniform visibility axiom, since they reserve the terminology visibility axiom for a weaker property in which the number R depends on the point $p\in\tilde{M}$. But, in this text, we only consider the property as defined above. R. Ruggiero proved the following important characterization for the compact case.

Proposition 2.3.1. *[Rug07, Theorem 6.8] Let M be a compact manifold without conjugate points. Then $\tilde{M}$ is a visibility manifold if and only if geodesic rays diverge and $\tilde{M}$ is Gromov hyperbolic.*

Since every Gromov hyperbolic space is quasi-convex, visibility manifolds satisfy the hypothesis of Theorem 2.2.1. Visibility manifolds were extensively studied by Eberlein in the 70s. One surprising result is that the visibility property does not depend on the metric.

Theorem 2.3.2. *[Ebe72, Theorem 5.1] Let (M, g) be a compact visibility manifold. Then, for any other metric g^* without conjugate points on M, (M, g^*) is a visibility manifold.*

We would like to state the following characterization for nonpositively curved manifolds.

Theorem 2.3.3. *[Ebe72, Theorem 4.1] Let M be a nonpositively curved compact manifold. Then M is a visibility manifold if and only if it does not contain an isometric totally geodesic immersed Euclidean plane $\mathbb{R}^2$. In particular, if $K < 0$ then M is a visibility manifold.*

Since compact surfaces of higher genus admit hyperbolic metrics, we have the following consequence.

Corollary 2.3.4. *Compact surfaces without conjugate points and genus higher than one are visibility manifolds.*

As a result, the conclusions of Theorem 2.2.1 hold for these surfaces.

2.4 Busemann functions and horospheres

We now turn our attention to Busemann functions. For $v \in T^1\tilde{M}$ and $t > 0$, consider the function $b_{v,t} : \tilde{M} \to \mathbb{R}$ defined by $b_{v,t}(x) = d(c_v(t), x) - t$. It is easy to show using the triangular inequality that these functions converge pointwise when t goes to infinity. The Busemann function $b_v : \tilde{M} \to \mathbb{R}$ is defined by

$$b_v(x) = \lim_{t \to +\infty} b_{v,t}(x).$$

Proposition 2.4.1. *[Kni86, 3.5 Satz] Let $\tilde{M}$ be a simply connected manifold without conjugate points. For every $v \in T^1\tilde{M}$, the Busemann function b_v is C^{1+Lip} (this means C^1 with locally Lipschitz derivatives). Moreover, $||\nabla b_v|| = 1$ and the integral curves of ∇b_v are geodesics.*

The following lemma expresses the relation between the geometry of spheres and stable solutions of the Jacobi equation.

Lemma 2.4.2. *Let $v \in T^1\tilde{M}$. The shape operator at the point $x \in \tilde{M}$ of the sphere centered at $c_v(t), t > 0$ and passing through x is given by*

$$-\nabla_X \nabla b_{v,t}(x) = J'_{-\nabla b_{v,t}(x), X, d(c_v(t), x)}(0).$$

Proof. We take $-\nabla b_{v,t}(x)$ as vector normal to the sphere at x. By definition, the shape operator with respect to the normal vector $-\nabla b_{v,t}(x)$ at a vector $X \in T_xM$ tangent to the sphere is

$$-\nabla_X \nabla b_{v,t}(x).$$

Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a curve in the sphere with $\alpha(0) = x$ and $\dot{\alpha}(0) = X$. Let $r = d(c_v(t), x)$ be the radius of the sphere. The curve α can be written as

$$\alpha(s) = \exp_{c_v(t)}(rV(s))$$

for a curve $V : (-\varepsilon, \varepsilon) \to T^1_{c_v(t)}M$. We define a variation of geodesics

$$f(u,s) = \exp_{c_v(t)}(uV(s)),$$

so that $\frac{\partial f}{\partial u}(r,s) = \nabla b_{v,t}(\alpha(s))$. Hence, if we consider the Jacobi field $J(u) = \frac{\partial f}{\partial s}(u,0)$ along $u \mapsto f(u,0)$, the shape operator is

$$-\frac{\partial^2 f}{\partial s \partial u}(r,0) = -J'(r).$$

This Jacobi field J satisfies $J(0) = 0$ and $J(r) = X$. The result follows by changing the origin along the geodesic from $c_v(t)$ to x. □

Proof of Proposition 2.4.1. The fact that b_v is C^1 and $\nabla b_{v,t}$ converges pointwise to ∇b_v goes back to J. H. Eschenburg [Esc77, Proposition 1].

Claim. Given a compact subset K of $\tilde{M}$, there exists a constant $L > 0$ such that, for all $v \in T^1\tilde{M}$, if t is large enough, $\nabla b_{v,t}$ is L-Lipschitz on K.

Proof. Let $\mathbf{B}_{v,s}$ denote the solution of the matrix Jacobi equation along c_v with $\mathbf{B}_{v,s}(0) = Id$ and $\mathbf{B}_{v,s}(s) = 0$. We consider the differentiable operator which sends $X \in v^\perp$ to $-\nabla_X \nabla b_{v,t}(x) \in v^\perp$. By Lemma 2.4.2, this operator in coordinates is given by the matrix

$$\mathbf{B}'_{-\nabla b_{v,t}(x), d(c_v(t),x)}(0).$$

We showed in the proof of Proposition 1.3.2 that, for $s > 0$, $\mathbf{B}'_{v,s}(0)$ is increasing and bounded by $\mathbf{B}'_{v,-1}(0)$. Hence, if $d(c_v(t), x) \geq 1$,

$$\mathbf{B}'_{-\nabla b_{v,t}(x),1}(0) \leq \mathbf{B}'_{-\nabla b_{v,t}(x),d(c_v(t),x)}(0) \leq \mathbf{B}'_{-\nabla b_{v,t}(x),-1}(0)$$

Since K is compact, if t is large enough, $d(c_v(t), x) \geq 1$ for all $x \in K$. Moreover, $\mathbf{B}'_{w,1}(0)$ and $\mathbf{B}'_{w,-1}(0)$ are continuous on w. Let L be a bound of their norm for $w \in T^1K$. We conclude that for all $v \in T^1M$, if t is large enough the norm of $\mathbf{B}'_{-\nabla b_{v,t}(x),d(c_v(t),x)}(0)$ is bounded by L. This shows that $\nabla b_{v,t}$ is L-Lipschitz on K. □

Now we can apply Arzelà-Ascoli to $\nabla b_{v,t}$. Since $\nabla b_{v,t}$ already converges pointwise to ∇b_v, we deduce that the convergence is uniform. Moreover, the limit ∇b_v is also L-Lipschitz on K.

□

We deduce from the proof that if the curvature of $\tilde{M}$ is bounded from below by a constant $-b^2$, ∇b_v is globally L-Lipschitz, where L depends only on b.

The level sets of a Busemann function

$$b_v^{-1}(k) \subset \tilde{M},\ k \in \mathbb{R}$$

are called horospheres and are C^1 submanifolds of dimension $n-1$. We observe that for all $s \in \mathbb{R}$, $b_{g_s v} = b_v + s$ so $b_{g_s v}$ and b_v define the same horospheres. The set $b_v^{-1}(k)$ can be obtained as the image of $b_v^{-1}(0)$ by the integral flow of ∇b_v at time k. This also shows that horospheres of the same Busemann function are equidistant. We will usually prefer to consider lifts of these hypersurfaces to the unit tangent bundle.

Definition 2.4.1. The stable horosphere of $v \in T^1\tilde{M}$ is the subset

$$\tilde{H}^s(v) := \{ -\nabla b_v(x) \,|\, x \in b_v^{-1}(0)\}.$$

The unstable horosphere of v is

$$\tilde{H}^u(v) := \{ \nabla b_{-v}(x) \,|\, x \in b_{-v}^{-1}(0)\}.$$

These two spaces satisfy $\tilde{H}^s(v) = -\tilde{H}^u(-v)$. They are Lipschitz submanifolds of $T^1\tilde{M}$, each of dimension $n-1$. Since the displacement of a horosphere in $\tilde{M}$ on its perpendicular geodesics produces another horosphere, we have

$$g_t\tilde{H}^s(v) = \tilde{H}^s(g_t v),$$

so we can say that the geodesic flow preserves the horospheres. Finally, we observe that Busemann functions are invariant by the isometries of $\tilde{M}$, so horospheres are also invariant by the isometries. This allows us to define horospheres on non simply connected manifolds as the images of horospheres by the covering map.

Proposition 2.4.3. *[Kni86, 3.8 Satz] Let $\tilde{M}$ be a simply connected manifold without conjugate points. Assume that the stable Green bundles $v \mapsto G^s(v)$ are continuous. Then, the stable horospheres $H^s(v)$ of $T^1\tilde{M}$ are C^1 manifolds, which depend continuously on v in the C^1 topology, and their tangent spaces $T_vH^s(v)$ are the stable Green spaces $G^s(v)$.*

Consequently, the stable horospheres form a continuous g_t-invariant foliation with C^1 leaves if the Green bundles are continuous. Without this hypothesis we do not even know if the map $v \mapsto H^s(v)$ is continuous. To our knowledge, the most general results involving the continuity of the horospheres for compact manifolds are the following. We remark that the continuity of the horospheres $\tilde{H}^s(v)$ in v is equivalent to the continuity of $v \mapsto b_v$ in the C^1 topology.

Theorem 2.4.4. *Let M be a complete manifold without conjugate points. If the universal covering $\tilde{M}$ is quasi-convex and geodesic rays diverge, then the horospheres $H^s(v)$ depend continuously on v.*

Proof. Let $v_n \to v \in T^1\tilde{M}$ and fix a compact K of $\tilde{M}$. We can assume that v is in the interior of K by extending K if necessary. We want to show that b_{v_n} and ∇b_{v_n} converge uniformly to b_v and ∇b_v on K. We observe that $-\nabla b_{v,t}(x) = V(x, c_v(t))$ and the equality extends to the limit $t \to +\infty$, since V is continuous (Theorem 2.2.1) and $\nabla b_{v,t} \to \nabla b_v$ as seen in the proof of Theorem 2.4.1, so

$$-\nabla b_v(x) = V(x, v_+).$$

Now, for all $x \in \tilde{M}$ we see that $\nabla b_{v_n}(x) = -V(x, v_{n+}) \to -V(x, v_+) = \nabla b_v(x)$ when $n \to \infty$. Moreover, the convergence is uniform for $x \in K$. If this was not the case, we could find $\varepsilon > 0$, a subsequence $n_k \to +\infty$ and points $x_k \in K$ such that

$$\angle(V(x_k, v_{n_k+}), V(x_k, v_+)) \geq \varepsilon > 0.$$

But letting w be an accumulation point of $V(x_k, v_{n_k+})$, we would have $w_+ = \lim v_{n_k+} = v_+$ and $x_k \to \pi(w)$, and the last equation would give in the limit $\angle(V(\pi(w), v_+, V(\pi(w), v_+)) \geq \varepsilon$, which is clearly a contradiction.

From the inequality $|b_w(x) - b_w(y)| \leq d(x,y)$, we see that the family of functions $\{b_{v_n}|_K\}_n$ is equicontinuous and uniformly bounded, so by Arzelà-Ascoli it is relatively compact in the uniform topology. Let $f = \lim b_{v_{n_k}}$ be one of its accumulation points. Since the pair $(b_{v_{n_k}}, \nabla b_{v_{n_k}})$ converges uniformly to $(f, \nabla b_v)$ on K, the function f is differentiable and $\nabla f = \nabla b_v$. Therefore there exists a constant λ such that $f = b_v + \lambda$. But $f(\pi(v)) = \lim b_{v_{n_k}}(\pi(v_{n_k})) = 0$, so $\lambda = 0$. The unique accumulation point of $\{b_{v_n}|_K\}_n$ is b_v so in fact b_{v_n} has to converge to b_v on K. We have proved that b_{v_n} converges to b_v in the C^1 topology, which is equivalent to the continuity of the horospheres. □

In the converse direction we have the following theorem.

Theorem 2.4.5. *[Rug03, Theorem 1] Let M be a compact manifold without conjugate points. If the horospheres $H^s(v)$ depend continuously on v, then geodesic rays diverge uniformly.*

Next we state the relation between asymptotic geodesics and the orbits of the gradient of a Busemann function, something which is already implicit in the ideas of the Theorem 2.4.4.

Lemma 2.4.6. *Let $\tilde{M}$ be quasi-convex and assume that geodesic rays diverge. Then the integral curves of the gradient ∇b_v of a Busemann function b_v are geodesics asymptotic to c_v. Conversely, is c_w is positively asymptotic to c_v (that is $v_+ = w_+$), then c_w is an orbit of ∇b_v. In fact, $b_v = b_w + b_v(\pi(w))$.*

Thanks to the previous lemma, we can speak of a horosphere centered at $\xi \in \partial\tilde{M}$, which is a level set of any Busemann function b_v with $v \in T^1\tilde{M}$ pointing at ξ. Fixing the vector v, notice that there is a bijection from $\mathbb{R}$ to the set of horospheres centered at ξ given by $t \mapsto b_v^{-1}(t)$.

Definition 2.4.2. The weak stable manifold of $v \in T^1\tilde{M}$ is the subset

$$\tilde{W}^{ws}(v) := \cup_{t\in\mathbb{R}} \tilde{H}^s(g_t v) = \{\, -\nabla b_v(x) \,|\, x \in \tilde{M}\}.$$

The weak unstable manifold of v is

$$\tilde{W}^{wu}(v) := \cup_{t\in\mathbb{R}} \tilde{H}^u(g_t v) = \{\, \nabla b_{-v}(x) \,|\, x \in \tilde{M}\}.$$

We define the weak stable and unstable manifolds on T^1M as the projections of the weak stable and unstable manifolds on $T^1\tilde{M}$.

If $\tilde{M}$ is quasi-convex and geodesic rays diverge, by Lemma 2.4.6, the weak stable and unstable manifolds can also be characterized as

$$\tilde{W}^{ws}(v) = \{w \in T^1\tilde{M} \,|\, w_+ = v_+\}, \quad \tilde{W}^{wu}(v) = \{w \in T^1\tilde{M} \,|\, w_- = v_-\}.$$

Let $\tilde{M}$ be a quasi-convex manifold without conjugate points, on which geodesic rays diverge uniformly. For $\xi \in \partial\tilde{M}$ and $x, y \in \tilde{M}$, we introduce the notation

$$\beta_\xi(x,y) := b_{V(y,\xi)}(x),$$

that we will use later. Notice that this quantity does not depend on x and y, but on the horospheres centered at ξ containing them. The function β_ξ is called a Busemann cocycle. By Theorem 2.4.1 and Theorem 2.4.4, $\beta_\xi(x,y)$ depends continuously on the triple $(\xi, x, y) \in \partial\tilde{M} \times \tilde{M} \times \tilde{M}$. We extend the notation by putting that $\beta_z(x,y) = d(x,z) - d(y,z)$ if $(z,x,y) \in \tilde{M} \times \tilde{M} \times \tilde{M}$.

Lemma 2.4.7. *Let $\tilde{M}$ be a simply connected quasi-convex manifold without conjugate points where geodesic rays diverge. Then the function*

$$\begin{array}{ccc} \bar{M} \times \tilde{M} \times \tilde{M} & \longrightarrow & \mathbb{R} \\ (z, x, y) & \longmapsto & \beta_z(x, y) \end{array}$$

is continuous.

2.5 Strips and endpoints of geodesics

We would like to describe the unit tangent bundle of a simply connected manifold $\tilde{M}$ without conjugate points using only points on the boundary at infinity. In this section we explain how this is possible. Recall that we associate to each vector $v \in T^1\tilde{M}$ two natural points at infinity v_- and $v_+ \in \tilde{M}$, which are respectively the asymptotic classes of the negative and positive geodesic rays generated by v. These points have to be different because geodesics on $\tilde{M}$ are globally minimizing. We consider the map

$$\begin{array}{cccc} P: & T^1\tilde{M} & \longrightarrow & \partial^2\tilde{M} \\ & v & \longmapsto & (v_-, v_+), \end{array}$$

where $\partial^2\tilde{M} = (\partial\tilde{M} \times \partial\tilde{M}) \setminus \Delta$ and Δ is the diagonal of $\partial\tilde{M} \times \partial\tilde{M}$.

All the vectors tangent to a given geodesic have the same image under P. We will see that there may be other vectors with this same image. We will also investigate in which situations P is surjective. In other terms, we want to know if, given two distinct points at infinity, there exists a geodesic joining them; and when this happens, if the geodesic is unique.

Proposition 2.5.1. *Let $\tilde{M}$ satisfy the visibility axiom. Then the map P is surjective.*

Proof. Let $\xi_1, \xi_2 \in \partial\tilde{M}$ be two distinct points, and consider the geodesic rays c_1, c_2 in each class starting at the same point $p \in \tilde{M}$. Let c_t be the geodesic joining $c_1(t)$ to $c_2(t)$. The angle $\angle_p(c_1(t), c_2(t))$ is constant for all t, so by the visibility property there exists a constant R such that $d(c_t, p) \leq R$ for all $t > 0$. We can parameterize c_t such that $d(c_t(0), p) \leq R$. By compactness, when t goes to infinity, c_t must accumulate to some geodesic c. This geodesic will be asymptotic to c_1 in one direction, and to c_2 in the other. In other words, c joins ξ_1 to ξ_2. □

Without the hyperbolicity provided by the visibility hypothesis we cannot expect P to be surjective. For example, on the flat plane $\mathbb{R}^2$ we can only join antipodal points.

Now we turn to the question of vectors whose images under P are equal. Given $v \in \tilde{M}$, the set

$$\tilde{S}(v) := \{w \in T^1\tilde{M} \mid \sup_{t\in\mathbb{R}} d(c_v(t), c_w(t)) < +\infty\} = P^{-1}(v_-, v_+)$$

is called the strip of v. We observe that its projection to $T^1\tilde{M}$ is

$$\pi(\tilde{S}(v)) = \{c(\mathbb{R}) \mid c \text{ is a geodesic with } d_H(c, c_v) < +\infty\},$$

a maximal set of biasymptotic geodesics.

Proposition 2.5.2. *[RR21, Lemma 3.1]Let M be a compact manifolds without conjugate points such that $\tilde{M}$ is quasi-convex and geodesic rays diverge on $\tilde{M}$. For every $v \in T^1\tilde{M}$ and $w \in \tilde{S}(v)$, there exists a connected compact subset $\Sigma(v,w)$ of $\tilde{H}^s(v) \cap \tilde{H}^u(v)$ which contains v and*

$$w \in \tilde{S}(v,w) := \bigcup_{t\in\mathbb{R}} g_t\Sigma(v,w) \subset \tilde{S}(v).$$

[Pes77, Theorem 7.3] If $\tilde{M}$ has no focal points, we can choose $\Sigma(v,w)$ to be of the form

$$\Sigma(v,w) = \{\nabla b_v(\alpha(s)) \mid s \in [0,l]\},$$

where $\alpha : [0,l] \to \tilde{M}, l \geq 0$ is a unit speed geodesic. Moreover, the set $\pi(\tilde{S}(v,w))$ is isometric to $[0,l] \times \mathbb{R}$ via the map $(s,t) \mapsto c_{\nabla b_v(\alpha(s))}(t)$.

We introduce the following notations for the intersections between stable and unstable horospheres.

Definition 2.5.1. For $v \in T^1\tilde{M}$, we define

$$\tilde{I}(v) = \tilde{H}^s(v) \cap \tilde{H}^u(v).$$

For $v \in T^1M$, we consider any lift $\tilde{v}$ of v to $T^1\tilde{M}$ and we define

$$I(v) = d\pi(\tilde{I}(v)).$$

The set $\tilde{I}(v)$ $(I(v))$ will be said to be trivial if $\tilde{I}(v) = \{v\}$ $(I(v) = \{v\})$. The expansive set $\mathcal{E}$ is the set of vectors $v \in T^1\tilde{M}$ for which $\tilde{I}(v)$ is trivial,

$$\mathcal{E} := \{v \in T^1\tilde{M} \mid \tilde{I}(v) = \{v\}\}.$$

If M is a surface, we refer to $\tilde{I}(v)$ or $I(v)$ as the interval of v.

Proposition 2.5.2 implies that the set $\tilde{S}(v)$ is homeomorphic to $\tilde{I}(v) \times \mathbb{R}$. In other words, the appearence of biasymptotic geodesics is caused by nontrivial intersections of the stable horosphere with the unstable one. If $\tilde{M}$ is a surface, we say that the set $\tilde{I}(v)$ is an interval because it is homeomorphic to an interval of $\mathbb{R}$, since $\tilde{H}^s(v)$ is 1-dimensional.

We observe that $\mathcal{E}$ is g_t-invariant, in the sense that $g_t(\mathcal{E}) = \mathcal{E}$ for all $t \in \mathbb{R}$. In order to study the dynamics of g_t, it is crucial to understand the set expansive set $\mathcal{E}$ and determine how big it is. This can be done in various situations, as we will see later.

If M is a visibility axiom, we can control the size of each nontrivial intersection $\tilde{I}(v)$.

Proposition 2.5.3. *Let $\tilde{M}$ satisfy the visibility axiom. Then there exists a constant $Q > 0$ such that, for all $v \in T^1\tilde{M}$, the diameter of $\tilde{I}(v)$ is bounded by Q.*

Proof. Let c be a geodesic biasymptotic to the geodesic generated by $v \in T^1\tilde{M}$. For $n \geq 0$, consider the vectors v_n and $w_n \in T^1_{\pi(v)}\tilde{M}$ pointing to $c(n)$ and $c(-n)$ respectively. Because of the continuity of V, we have $\lim v_n = v$ and $\lim w_n = -v$. Since the angle formed by $c|_{[-n,n]}$ from $\pi(v)$ does not go to 0, by the visibility property, there exists a constant $Q' > 0$ such that $d(\pi(v), c|_{[-n,n]}) \leq Q'$. This implies also that $d(\pi(v), c) \leq Q'$, and therefore Q' bounds the distance between any two biasymptotic geodesics. We can conclude that there exists a constant $Q > 0$ depending on Q' such that, for all $v \in T^1\tilde{M}$, $\operatorname{diam}\tilde{I}(v) \leq Q$. □

Without the visibility axiom we know little about the image of the map P. In the next section we will introduce the concept of rank 1 manifold. With this condition, if M is nonpositively curved or has no focal points, there are still strong results about P which ensure that certain points at infinity are joined by a geodesic.

2.6 Rank 1 manifolds

We have seen how the behaviour of the Green subbundles has a strong impact on the structure of the manifold and the dynamics of the geodesic flow. For example, Eberlein's Theorem 1.3.8 establishes that the linear independence of the Green subbundles implies the Anosov property of the geodesic flow. In a general manifold without conjugate points, the vectors whose stable and unstable Green subspaces are linear independent are said to have rank 1. Thus, the rank 1 set is

$$R_1 := \{v \in T^1M \mid G^u(v) \cap G^s(v) = \{0\}\}.$$

This set is g_t-invariant. It is immediate that if $\tilde{M}$ has continuous Green bundles, the rank 1 set R_1 is open and

$$R_1 \subset \mathcal{E}$$

because horospheres are tangent to the Green bundles (see Theorem 2.4.3), so they intersect trivially on R_1. If we do not make any additional assumptions on the manifold, there is no reason why R_1 should be nonempty.

Definition 2.6.1. A manifold M without conjugate points has rank 1 if $R_1 \neq \emptyset$.

The concept of rank takes its full meaning for manifolds with nonpositive curvature or with no focal points.

Lemma 2.6.1. *Let $\tilde{M}$ be a manifold without focal points. Then, for every $v \in T^1\tilde{M}$ the dimension of the space of parallel Jacobi fields along the geodesic generated by v is equal to*

$$1 + \dim(G^s(v) \cap G^u(v)). \tag{2.3}$$

Proof. The space of parallel Jacobi fields along v decomposes as $N \oplus \mathbb{R}\,\dot{c}_v$, where N is the space of perpendicular parallel Jacobi fields. We want to show that N is the set of Jacobi fields which are both stable and unstable.

($\subset$) If $J \in N$, $J' = 0$ so $||J||^{2'} = 2\langle J, J'\rangle = 0$ so $||J||$ is constant. By 1.3.10, Jacobi fields bounded for positive time are stable, and for negative time, unstable, so J is both stable and unstable.

($\supset$) Since $\tilde{M}$, has no focal points, the norm of stable Jacobi fields is non-decreasing, and the norm of unstable Jacobi fields is non-decreasing. So if J is both stable and unstable, the norm of J is constant.

Recall from the proof of Theorem 1.3.2, that fixing a parallel orthonormal frame along c_v, a perpendicular stable Jacobi field J can be written in coordinates as $\mathbf{B}^s(t)x$ for some $x \in \mathbb{R}^{n-1}$, where $\mathbf{B}^s$ is the stable solution of the matrix Jacobi equation with $\mathbf{B}^s(0) = Id$ and x represents the initial condition $J(0)$. We also recall that $\mathbf{B}^s(t)$ is nonsingular for all $t \in \mathbb{R}$ and that $\mathbf{U}(t) = \mathbf{B}^{s\prime}(t)(\mathbf{B}^s(t))^{-1}$ is a symmetric solution of the Ricatti equation.

Now, for all $x \in \mathbb{R}^{n-1}$ and all $t \in \mathbb{R}$, denoting $y = (\mathbf{B}^s(t))^{-1}x$ we have

$$\langle \mathbf{U}(t)x, x\rangle = \langle \mathbf{B^s}'(t)y, \mathbf{B}^s(t)y\rangle = \frac{1}{2}\frac{d}{ds}|_{s=t}\langle \mathbf{B}^s(s)y, \mathbf{B}^s(s)y\rangle = \frac{1}{2}(||\mathbf{B}^s y||^2)'(t) \leq 0,$$

since the norm of every stable Jacobi field is non-decreasing. This shows that $\mathbf{U}$ is negative semidefinite.

If $t \mapsto \mathbf{B}^s(t)y$ represents a unstable Jacobi field, then the computation above implies that for all $t \in \mathbb{R}$,

$$0 = \langle \mathbf{B^s}'(t)y, \mathbf{B}^s(t)y\rangle = \langle \mathbf{U}(t)\mathbf{B}^s(t)y, \mathbf{B}^s(t)y\rangle,$$

so $\mathbf{U}(t)\mathbf{B}^s(t)y = 0$ because $\mathbf{U}(t)$ is semidefinite. This says that $\mathbf{B^s}'(t)y = 0$ so the Jacobi field $t \mapsto \mathbf{B}^s(t)y$ is parallel. □

The lemma provides a way to define the rank of the vector v for manifolds with no focal points which is consistent with the definition of rank 1 vector made at the beginning of the section.

Rank 1 vectors behave like in a hyperbolic space in many senses, the following results clarifies the structure of the rank 1 set in terms of the boundary of the manifold. It was first proved by W. Ballmann in nonpositive curvature [Bal82, Lemma 3.1].

Proposition 2.6.2. *[LWW20, Proposition 7] Let $\tilde{M}$ be a manifold without focal points and let $v \in T_1\tilde{M}$ be a rank 1 vector. For every $\varepsilon > 0$, there exist neighborhoods U, V of v_-, v_+, respectively, in $\partial\tilde{M}$ such that for every $(\eta, \xi) \in U \times V$, there exists a rank 1 vector w with $w_- = \eta$ and $w_+ = \xi$ with*

$$d(v, w) < \varepsilon.$$

In particular, $\tilde{S}(w)$ is just the orbit of w by g_t.

This result improves our understanding of the structure of the unit tangent bundle of the universal cover. Let $0 \in \tilde{M}$ be a reference point. We consider the continuous map

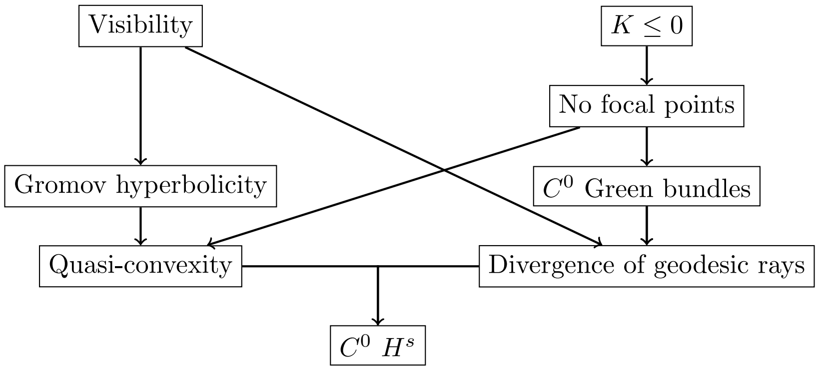
$$\begin{array}{cccc} \bar{P}: & T^1\tilde{M} & \longrightarrow & \partial^2\tilde{M} \times \mathbb{R} \\ & v & \longmapsto & (v_-, v_+, \beta_{v_-}(0, \pi(v)). \end{array}$$

Proposition 2.6.2 directly implies.

Corollary 2.6.3. *The restriction of the map $\bar{P}$ to the rank 1 set $\bar{P} : R_1 \to \bar{P}(R_1)$ is a homeomorphism.*

2.7 Diagram

To finish the chapter, we summarize the different classes of manifolds without conjugate points in the following diagram.



Part II

Ergodic properties of horospheres

Chapter 3

Generalizations of the Patterson-Sullivan measure

In this chapter we construct a series of objects of measure-theoretic nature that we will use later in the study of the dynamics of horospheres. These methods were pioneered by Patterson and Sullivan [Pat76, Sul79] for hyperbolic spaces, and then extended to more general settings [Yue96, Rob03, Kni02, LP16, CKW21]. The basic idea is to exploit the duality between the geodesic flow on a given space and the action of the covering transformations group on the universal cover. The boundary at infinity and the description of the unit tangent bundle of the universal cover using this boundary play a crucial role in the theory.

In Section 3.1 we introduce the limit set and establish the relation with the dynamics of the geodesic flow. The first important object we consider is the Patterson-Sullivan measure, which is a measure supported on the boundary at infinity which contains a lot of information about the dynamics on the quotient as we will explain in Section 3.2. From this measure, in Section 3.3, we construct a measure invariant by the geodesic flow, which we will call the Bowen-Margulis measure. This measure is proved to be the unique measure of maximal entropy in many situations. In the final section, 3.4, we define a family of measures supported on the horospheres.

3.1 Limit set and nonwandering subset of the geodesic flow

Let $\tilde{M}$ be a quasi-convex simply connected manifold without conjugate points with divergence of geodesic rays. We will consider a group of isometries Γ of $\tilde{M}$. The group Γ is called discrete if it is discrete in the compact-open topology. For example, if $\tilde{M}$ is the universal cover of a manifold M, then the group of covering transformations of the covering $\Pi : \tilde{M} \to M$ is a discrete group of isometries. In this case, the action is also free. We do not need to assume that Γ acts freely yet, so for now $\tilde{M}/\Gamma$ is not necessarily a manifold.

We can extend the action of Γ on $\tilde{M}$ to $\bar{M}$ because isometries preserve asymptotic classes. Moreover, each $\gamma \in \Gamma$ is a homeomorphism of $\bar{M}$. We will often consider the orbit $\Gamma x \subset \bar{M}$ of a point $x \in \tilde{M}$. This orbit can only accumulate on the boundary $\partial\tilde{M}$. Moreover, if $\xi = \lim \gamma_n x$, then for every $y \in \tilde{M}$, $\lim \gamma_n y = \xi$. This is because $\angle_x(\gamma_n y, \gamma_n x) \to 0$ by the uniform divergence of geodesic rays, so $\angle_x(\gamma_n y, \xi) \to 0$ since $\angle_x(\gamma_n x, \xi) \to 0$. This shows that the set of accumulation points of an orbit Γx does not depend on the point x.

Definition 3.1.1. The limit set Λ of Γ is defined as the set of accumulation points of an orbit Γx, i.e.
$$\Lambda = \overline{\Gamma x} \cap \partial \tilde{M}.$$

An isometry γ is called axial if there exists a geodesic $c : \mathbb{R} \to \tilde{M}$ and there exists $T > 0$ such that for all $t \in \mathbb{R}$, $\gamma c(t) = c(t+T)$. The geodesic c is called an axis of γ. If c is an axis of an isometry γ and we denote $v = \dot{c}(0)$, then $v_+ = \lim \gamma^n c(0)$ and $v_- = \lim \gamma^{-n} c(0)$ are in the limit set Λ. We remark that, when $M = \tilde{M}/\Gamma$ is a manifold, the axes of axial isometries are exactly the lifts of closed geodesics of M. An axial isometry γ is said to have rank 1 if it has an axis which is a rank 1 geodesic.

At this point the theories of the two cases that we are mainly interested in differ a bit. To our knowledge there is not a theory which unifies the study of discrete isometry groups of nonpositively curved rank 1 manifolds and of visibility manifolds. In the next theorem we summarize the main results that will be needed in the sequel. The proof in nonpositive curvature is due to W. Ballmann and for visibility manifolds, which are hyperbolic spaces, to M. Gromov.

Proposition 3.1.1. *[Bal82, Theorem 2.8] [Gro87, Lemma 8.2.A] Let $\tilde{M}$ be a simply connected manifold without conjugate points and let Γ be a discrete group of isometries of $\tilde{M}$ which satisfy one of the following:*

A) $\tilde{M}$ is nonpositively curved and Γ contains a rank 1 axial isometry,

B) $\tilde{M}$ is a visibility manifold and Γ contains an axial isometry.

Then the limit set Λ of Γ has either 2 or infinitely many elements. In the latter case the limit set Λ is minimal under Γ, that is, for every $\xi \in \Lambda$, $\overline{\Gamma\xi} = \Lambda$.

Both in case A) and B), Γ is said to be non-elementary when Λ has infinitely many elements.

Recall that a vector $v \in T^1M$ is said to be nonwandering for the flow g_t if for any neighborhood U of v there exists $t > 1$ such that $g_tU \cap U \neq \emptyset$. Alternatively we can say that there exists a sequence of vectors $v_n \to v$ and a sequence of times $t_n \to +\infty$ such that $g_{t_n} v_n \to v$. We will denote by Ω the set of nonwandering points of g_t on T^1M and by $\tilde{\Omega}$ its lift to $T^1\tilde{M}$, so
$$\tilde{\Omega} = \{v \in T^1\tilde{M} \mid \exists\, v_n \to v,\ t_n \to +\infty,\ \gamma_n \in \Gamma \text{ s.t. } \gamma_n g_{t_n} v_n \to v\}.$$

Proposition 3.1.2. *Let $\tilde{M}$ be a simply connected quasi-convex manifold without conjugate points where geodesic rays diverge. Let Γ be a discrete group of isometries. Assume that the limit set Λ is infinite and that Γ acts minimally on Λ. A vector $v \in T^1\tilde{M}$ is in $\tilde{\Omega}$ if and only if v_- and v_+ belong to Λ.*

Proof. Let $v \in \tilde{\Omega}$. There exists a sequence of vectors $v_n \to v$, a sequence of times $t_n \to +\infty$ and a sequence of isometries $\gamma_n \in \Gamma$ such that $\gamma_n g_{t_n} v_n \to v$.

First we prove that $\pi(g_{t_n} v_n) \to v_+$, where $\pi : T^1\tilde{M} \to \tilde{M}$ is the projection to the base point. If this was not true, we could assume by compactness that $\pi(g_{t_n} v_n)$ has an accumulation point $z \in \bar{M}$ different from v_+, and this accumulation point has to be in $\partial\tilde{M}$. But, since
$$v_n = V(\pi(v_n), \pi(g_{t_n} v_n)) \to v = V(\pi(v), v_+),$$

and because of the continuity of the map V, we would get $V(\pi(v), v_+) = V(\pi(v), z)$, which contradicts that z is different from v_+.

We observe that $d(g_{t_n} v_n, \gamma_n^{-1} v) \to 0$, so $d(\pi(g_{t_n} v_n), \pi(\gamma_n^{-1} v)) \to 0$. By the uniform divergence of geodesic rays, given $\varepsilon > 0$ there exists $T > 0$ such that if $d(c_1(t), c_2(t)) \leq 1$ for $t \geq T$ and $c_1(0) = c_2(0) = x$ then $\angle_x(c_1(t), c_2(t)) \leq \varepsilon$. Applying this to the geodesics joining $\pi(v)$ to $\pi(g_{t_n} v_n)$ and $\pi(v)$ to $\pi(\gamma_n^{-1} v)$ we see that the angle $\angle_{\pi(v)}(\pi(g_{t_n} v_n), \pi(\gamma_n^{-1} v))$ tends to 0. We deduce that $\pi(\gamma_n^{-1} v)$ also tends to v_+. So v_+ is in the closure of $\Gamma\pi(v)$. A similar argument proves that v_- belongs to Λ.

Now we prove the converse. We will need the following lemma.

Lemma 3.1.3. *Let $\xi, \eta \in \Lambda$, and let $x \in \tilde{M}$. There exists a sequence of isometries $(\gamma_n)_n$ such that $\lim \gamma_n x = \xi$ and $\lim \gamma_n^{-1} x = \eta$.*

Proof. Let $\xi \in \Lambda$. Consider the set $A_\xi \subset \Lambda$ of points $\eta \in \partial\tilde{M}$ such that there exists $(\gamma_n)_n$ with $\lim \gamma_n x = \xi$ and $\lim \gamma_n^{-1} x = \eta$. This set is nonempty because of the compactness. It is also Γ-invariant, because if $\lim \gamma_n x = \xi$ and $\lim \gamma_n^{-1} x = \eta$ then $\lim \gamma_n \gamma^{-1} x = \xi$ and $\lim \gamma\gamma_n^{-1} x = \gamma\eta$. The first property is true thanks to the uniform divergence of geodesic rays.

So if we prove that A_ξ is closed, we will conclude that $A_\xi = \Lambda$ by minimality. Let η_n be a sequence of A_ξ converging to $\eta \in \partial\tilde{M}$. Let, for each n, $\gamma_{n,k}$ a sequence such that $\lim_k \gamma_{n,k} x = \xi$ and $\lim_k \gamma_{n,k}^{-1} x = \eta_n$. We set $w = V(x, \xi)$ and $u = V(x, \eta)$ and we consider cones of the type $T_{w,\varepsilon,R}$ and $T_{u,\varepsilon,R}$ around w and u. For each n, let k_n large enough such that

$$\angle_x(\gamma_{n,k_n} x, \xi) \leq 1/n,$$

$$\angle_x(\gamma_{n,k_n}^{-1} x, \eta_n) \leq 1/n$$

$$d(\gamma_{n,k_n} x, x) \geq n,$$

$$d(\gamma_{n,k_n}^{-1} x, x) \geq n.$$

Now the sequences $\gamma_{n,k_n} x$ and $\gamma_{n,k_n}^{-1} x$ converge to ξ and η, respectively, by construction (see Figure 3.1). So we have proved that $\eta \in A_\xi$ □

Applying the lemma to v_-, v_+ and $\pi(v)$ we obtain a sequence γ_n with $\gamma_n \pi(v) \to v_-$ and $\gamma_n^{-1} \pi(v) \to v_+$. Consider the vectors $w_n \in T^1_{\gamma_n \pi(v)} \tilde{M}$ pointing to $\pi(v)$. Writing $t_n = d(\gamma_n \pi(v), \pi(v))$, we have $v_n = g_{t_n} w_n \in T^1_{\pi(v)} \tilde{M}$. Moreover, $\angle(v, v_n) = \angle_{\pi(v)}(v_-, \gamma_n \pi(v)) \to 0$, so $v_n \to v$.

On the other hand, $\gamma_n^{-1} g_{-t_n} v_n = \gamma_n^{-1} w_n = V(\pi(v), \gamma_n^{-1} \pi(v)) \to v$. This proves that v is nonwandering. □

If M is a Riemannian manifold of finite volume, then the Lebesgue measure on $T^1 M$ is finite. By the Poincaré recurrence Theorem, Lebesgue almost every point is g_t-recurrent. In particular, recurrent points are dense in $T^1 M$, so every point is nonwandering. If the hypothesis of Proposition 3.1.2 are satisfied, this is equivalent to saying that the limit set Λ is the whole boundary $\partial\tilde{M}$.

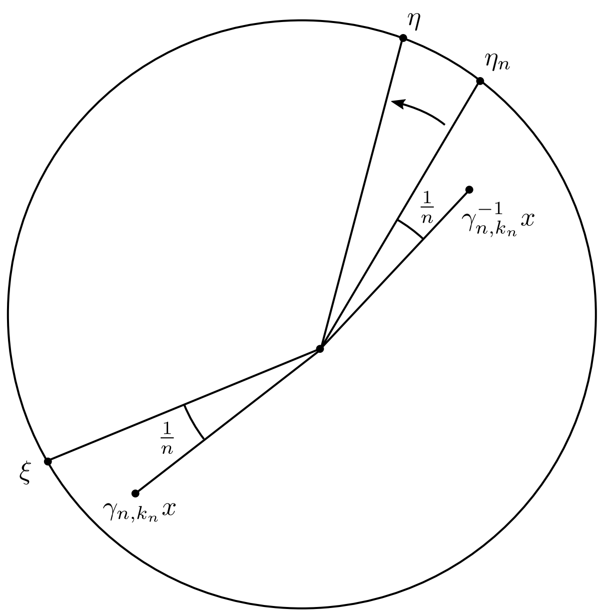


Figure 3.1: Proof of Lemma 3.1.3

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3.2 Patterson-Sullivan measure

As we said before, Patterson-Sullivan methods consist basically in extracting all the information of the action of Γ on the space $\tilde{M}$. The first interesting quantity is the growth rate of an orbit of Γ. We fix a point $0 \in \tilde{M}$ for the duration of the chapter.

Definition 3.2.1. Let Γ be a discrete group of isometries of a simply connected manifold $\tilde{M}$ without conjugate points. The critical exponent of Γ is defined as

$$\delta = \limsup_{R\to+\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma \mid d(0, \gamma 0) \leq R\}.$$

We remark that δ does not depend on the choice of the origin 0. If the curvature of $\tilde{M}$ is bounded below by a constant $-\kappa^2$, a volume comparison argument shows that δ is finite. When Γ is the covering group of a compact manifold M without conjugate points, A. Freire and R. Mañé established that the topological entropy of the geodesic flow of M is equal to the critical exponent.

Theorem 3.2.1. *[FMn82] Let M be a compact manifold without conjugate points and let Γ be the covering transformations group. Then the topological entropy of the geodesic flow $g_t : T^1M \to T^1M$ is equal to critical exponent δ of Γ.*

J. P. Otal and M. Peigné showed that the critical exponent and the topological entropy of the geodesic flow also coincide for non-elementary manifolds if the curvature is negatively pinched and has bounded derivatives [OP04].

Next we introduce the Poincaré series and justify the terminology used for δ.

Definition 3.2.2. Given two points $x, y \in \tilde{M}$, we define the Poincaré series of Γ depending on a real variable $s \in \mathbb{R}$ as

$$P(s; x, y) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)}.$$

The summability of the Poincaré series at each s is independent of x and y.

Proposition 3.2.2. *Let Γ be a discrete group of isometries of a simply connected manifold $\tilde{M}$ without conjugate points. Then the Poincaré series converges for $s > \delta$ and diverges for $s < \delta$.*

Proof. We write
$$N(R, x, y) = \#\{\gamma \in \Gamma \,|\, d(x, \gamma y) \leq R\},$$
which is sometimes called the orbital counting function. The set of distances $\{d(x, \gamma y) \,|\, \gamma \in \Gamma\}$ is discrete, so we can order them as a sequence $0 \leq t_1 < t_2 < \dots$. Given $R > 0$, we consider $i_0 \geq 0$ such that $t_{i_0} \leq R < t_{i_0+1}$, and we can write

$$\sum_{d(x,\gamma y)\leq R} e^{-sd(x,\gamma y)} = \sum_{i=1}^{i_0} \#\{\gamma \in \Gamma \,|\, d(x, \gamma y) = t_i\}\, e^{-st_i} = \sum_{i=1}^{i_0-1} \int_{t_i}^{t_{i+1}} N(t_i, x, y) e^{-st} s^{-1} \mathrm{d}t$$

$$+ N(t_{i_0}, x, y) e^{-st_{i_0}} = \int_0^R N(t, x, y) e^{-st} s^{-1} \mathrm{d}t + N(t_{i_0}, x, y) e^{-st_{i_0}}.$$

Applying the definition of δ we can control the growth of $N(t, x, y)$ and show that the previous expression converges for $s > \delta$ and diverges for $s < \delta$. □

The Poincaré series can have two different behaviours at $s = \delta$.

Definition 3.2.3. We say that Γ is divergent (resp. convergent), if the Poincaré series diverges (resp. converges) at $s = \delta$.

Now we will explain how to generalize the Patterson-Sullivan measure for quasi-convex manifolds with divergence of geodesic rays. This measure is constructed as a weak limit of measures supported on an orbit of Γ weighted by the Poincaré series. For technical reasons we need to consider a modified Poincaré series if Γ is of convergent. The following lemma explains how to obtain this modified series.

Lemma 3.2.3. *Let $(a_n)_n$ be a non-increasing sequence of positive numbers converging to 0 and consider a formal series $Q(s) = \sum_{n=1}^{+\infty} a_n^s$ depending on a real parameter $s \in \mathbb{R}$. Assume that Q has a critical exponent $\delta \geq 0$, that is, a number $\delta \geq 0$ for which $Q(s)$ converges for $s > \delta$ and diverges for $s < \delta$. Then, there exists a non-decreasing sequence $b_n \geq 1$ such that for all $\lambda > 0$, $\limsup(b_n/b_{n'})(a_n/a_{n'})^\lambda \leq 1$ when $n \geq n' \to +\infty$ and the modified series*

$$Q'(s) = \sum_{n=1}^{+\infty} b_n a_n^s$$

converges for $s > \delta$ and diverges for $s \leq \delta$.

Proof. If the series Q already diverges at δ, the statement is true without any modifications. If $\delta = 0$, then Q diverges at δ. So we can assume that $\delta > 0$. We choose a decreasing sequence ε_n of numbers in $(0, \delta)$ converging to 0. We set $m_1 = 1$ and $b_1 = 1$ and define recursively, a sequence $(m_k)_k$ of integers and the sequence $(b_n)_n$. Assume that $(m_k)_k$ is defined up to k and $(b_n)_n$ is defined up to m_k. Since Q diverges at $\delta - \varepsilon_k$, we can choose m_{k+1} such that

$$b_{m_k} a_{m_k}^{\varepsilon_k} \sum_{n=m_k+1}^{m_{k+1}} a_n^{\delta-\varepsilon_k} \geq 1.$$

For $m_k < n \leq m_{k+1}$ we set
$$b_n = b_{m_k} a_{m_k}^{\varepsilon_k} a_n^{-\varepsilon_k}.$$
The sequence b_n is non-decreasing because a_n is non-increasing. Now we have
$$\sum_{n=1}^{+\infty} b_n a_n^\delta = a_1^\delta + \sum_{k=1}^{+\infty} \sum_{n=m_k+1}^{m_{k+1}} b_{m_k} a_{m_k}^{\varepsilon_k} a_n^{\delta-\varepsilon_k} \geq a_1^\delta + \sum_{k=1}^{+\infty} 1 = +\infty.$$

From the definition of the b_n we can also see that, if $n \geq n' \geq m_k$, then
$$\frac{b_n}{b_{n'}} \leq \left(\frac{a_{n'}}{a_n}\right)^{\varepsilon_k}.$$
This implies the asymptotics of the statement. The convergence of Q for $s > \delta$ follows from the fact that $b_n = o(a_n^{-\lambda})$ for all $\lambda > 0$. □

We write the elements of the group Γ as a sequence γ_n such that $d(x, \gamma_n y)$ is non-decreasing. Applying the lemma to the Poincaré series $P(\,\cdot\,, x, y)$, we obtain coefficients b_γ with $\limsup (b_\gamma / b_{\gamma'}) e^{\lambda(d(x,\gamma' y) - d(x,\gamma y))} \leq 1$ when $d(x, \gamma y) \geq d(x, \gamma' y) \to +\infty$, for all $\lambda > 0$, and a modified Poincaré series
$$P'(s, x, y) = \sum_{\gamma \in \Gamma} b_\gamma e^{-sd(x,\gamma y)}.$$
We fix the coefficients b_γ, so we can forget the dependence on x and y. The summability of P' does not depend on x, y. Given $s > \delta$, and two points $x, y \in \tilde{M}$, since the Poincaré series converges at s, we can consider the measure on $\bar{M}$ defined by
$$\mu_{x,y,s} = \frac{\sum_{\gamma\in\Gamma} b_\gamma e^{-sd(x,\gamma y)} \delta_{\gamma y}}{P'(s, y, y)}. \tag{3.1}$$
We observe that the mass of $\mu_{x,y,s}$ satisfies
$$e^{-sd(x,y)} \leq ||\mu_{x,y,s}|| \leq e^{sd(x,y)}.$$
We endow the space of measures on $\bar{M}$ with the weak topology. Since $\bar{M}$ is compact, the unit ball of the space of measures is compact, so the sequence $\mu_{x,y,s}$ has at least a weak accumulation point when $s \to \delta^+$, and it is nonzero.

Proposition 3.2.4. *Let $\tilde{M}$ be a quasi-convex manifold without conjugate points where geodesic rays diverge. Let μ_{x,y,s_n} converge weakly to a measure μ_x of $\bar{M}$ on a sequence $s_n \to \delta^+$. Then:*

1. *The support of μ_x is contained in the limit set Λ.*
2. *For every $z \in \tilde{M}$, the sequence μ_{z,y,s_n} converges weakly to a measure μ_z, which is absolutely continuous with respect to μ_x and satisfies, for all $\xi \in \partial\tilde{M}$,*
$$\frac{\mathrm{d}\mu_z}{\mathrm{d}\mu_x}(\xi) = e^{-\delta\beta_\xi(z,x)}.$$
3. *The family of measures $\{\mu_z\}_{z\in\tilde{M}}$ is Γ-invariant, i.e. $\gamma_*\mu_z = \mu_{\gamma z}$.*

Proof. 1. Since μ_x is a weak limit of measures whose support is Γy, the support of μ_x must be contained in $\overline{\Gamma y} = \Gamma y \cup \Lambda$. But a point of the form γy cannot be contained in the support because, taking a small neighborhood U of γy, we have

$$\mu_{x,y,s_n}(U) = \frac{b_\gamma e^{-s_n d(x,\gamma y)}}{P'(s_n, x, y)} \to 0,$$

since the modified Poincaré series diverges at $s = \delta$. Hence, $\operatorname{supp} \mu_x \subset \Lambda$.

2. We observe that

$$\mathrm{d}\mu_{z,y,s_n}(p) = e^{-s(d(z,p)-d(x,p))} \mathrm{d}\mu_{x,y,s_n}(p).$$

Recall that $p \mapsto d(z,p) - d(x,p))$ extends to a continuous function on $\bar{M}$ denoted by $p \mapsto \beta_p(z,x)$ (Lemma 2.4.7). Taking the limit $n \to +\infty$ we get the desired result.

3. Let f be a continuous function on $\bar{M}$. The value of $\gamma_* \mu_x(f)$ is the limit of

$$\gamma_* \mu_{x,y,s_n}(f) = \frac{\sum_{\alpha \in \Gamma} b_{\gamma^{-1}\alpha} e^{-s_n d(\gamma x, \alpha y)} f(\alpha y)}{P'(s_n, y, y)}$$

and the value of $\mu_{\gamma x}(f)$ is the limit of

$$\mu_{\gamma x,y,s_n}(f) = \frac{\sum_{\alpha \in \Gamma} b_\alpha e^{-s_n d(\gamma x, \alpha y)} f(\alpha y)}{P'(s_n, y, y)}.$$

Thanks to the asymptotics of the coefficients b_α we have, for all $\lambda > 0$

$$e^{-\lambda d(y,\gamma y)} \leq \frac{b_{\gamma^{-1}\alpha}}{b_\alpha} \leq e^{\lambda d(y,\gamma y)}$$

for all $\alpha \in \Gamma$ but a finite number. Since the denominators are equal and tend to $+\infty$, the two expressions coincide in the limit.

□

Definition 3.2.4. A measure μ_x supported on $\partial\tilde{M}$ which is a weak accumulation point of the $\mu_{x,y,s}$ is called a Patterson-Sullivan measure. A conformal density of dimension $\alpha \geq 0$ for a group Γ is a family of measures $\{\sigma_x\}_{x\in\tilde{M}}$ supported on Λ satisfying

$$\forall x \in \tilde{M}, \forall \gamma \in \Gamma, \quad \gamma_* \sigma_x = \sigma_{\gamma x},$$

$$\forall x, y \in \tilde{M}, \forall \xi \in \partial\tilde{M}, \quad \frac{\mathrm{d}\sigma_y}{\mathrm{d}\sigma_x}(\xi) = e^{-\alpha \beta_\xi(y,x)}.$$

Proposition 3.2.4 implies that there always exists a conformal density of dimension δ. In the case that Γ acts minimally on Λ, the support of a conformal density is the whole limit set Λ. In the two situations under investigation, the critical exponent is positive in the non-elementary case.

Proposition 3.2.5. *[LP16, Coo93] Let $\tilde{M}$ be a simply connected manifold without conjugate points and let Γ be a discrete group of isometries of $\tilde{M}$ which satisfy one of the following:*

A) $\tilde{M}$ is nonpositively curved and Γ contains a rank 1 axial isometry,

B) $\tilde{M}$ is a visibility manifold and Γ contains an axial isometry.

Assume that Λ has infinitely many points. Then $\partial\tilde{M}$ does not support a Γ-invariant measure. In particular, the critical exponent δ is positive.

3.3 Bowen-Margulis measure

The next step is to define a measure on the unit tangent bundle T^1M invariant by the geodesic flow g_t called the Bowen-Margulis measure. This measure will be first constructed on $T^1\tilde{M}$ using Patterson-Sullivan measures.

Let $\tilde{M}$ be a quasi-convex simply connected manifold without conjugate points with divergence of geodesic rays. Recall that we defined the set $\partial^2\tilde{M}$ of pairs of distinct points at infinity and a map $P : T^1\tilde{M} \to \partial^2\tilde{M}$ sending a vector v to its endpoints (v_-, v_+). The set of geodesic endpoints is

$$E(\tilde{M}) := P(T^1\tilde{M}) = \{(v_-, v_+) \in \partial^2\tilde{M} \mid v \in T^1\tilde{M}\} \subset \partial^2\tilde{M}.$$

Unless $\tilde{M}$ satisfies the visibility axiom, the set $E(\tilde{M})$ is not necessarily equal to $\partial^2\tilde{M}$.

Definition 3.3.1. For $(\eta, \xi) \in E(\tilde{M})$, let $p_{\eta,\xi} \in \tilde{M}$ be any point on a geodesic joining η to ξ and define the Gromov product of ξ and η at $x \in \tilde{M}$ as

$$\langle \eta, \xi \rangle_x := \frac{1}{2}(\beta_\xi(x, p_{\eta,\xi}) + \beta_\eta(x, p_{\eta,\xi})).$$

Let us check that the previous definition does not depend on the choice of $p := p_{\eta,\xi}$. Let $p' \in \tilde{M}$ be another point in a geodesic joining η to ξ. Let v be the vector with base point at p pointing to ξ. By Proposition 2.5.2, $p' \in b_v^{-1}(t) \cap b_{-v}^{-1}(-t)$ for some $t \in \mathbb{R}$. Then, by the definition of the Busemann cocycle, we obtain

$$\begin{aligned} \beta_\xi(x, p) + \beta_\eta(x, p) &= \beta_\xi(x, p') + \beta_\xi(p', p) + \beta_\eta(x, p') + \beta_\eta(p', p) \\ &= \beta_\xi(x, p') + \beta_\eta(x, p') + b_v(p') + b_{-v}(p') = \beta_\xi(x, p') + \beta_\eta(x, p'). \end{aligned}$$

We next define an auxiliary measure on the set of geodesic endpoint pairs.

Definition 3.3.2. Let $\tilde{M}$ be quasi-convex and assume that geodesic rays diverge. Let Γ be a discrete group of isometries and let $\{\sigma_x\}_{x\in\tilde{M}}$ be a conformal density of dimension $\alpha \geq 0$. Fix $x \in \tilde{M}$. We define a Borel measure $\bar{\mu}$ on the set of geodesic endpoint pairs $E(\tilde{M})$ by its density

$$\mathrm{d}\bar{\mu}(\eta, \xi) = e^{2\alpha\langle \eta,\xi\rangle_x} \mathrm{d}\sigma_x(\xi)\, \mathrm{d}\sigma_x(\eta).$$

Proposition 3.3.1. *The measure $\bar{\mu}$ defined above does not depend on the choice of x and it is invariant by the diagonal action of Γ on $E(\tilde{M})$.*

Proof. If $y \in \tilde{M}$, we have

$$2\langle \eta, \xi \rangle_y = \beta_\eta(y, x) + \beta_\xi(y, x) + 2\langle \eta, \xi \rangle_x$$

and $\mathrm{d}\sigma_y(\xi) = e^{-\alpha\beta_\xi(y,x)} \mathrm{d}\sigma_y(\xi)$. Hence, $\bar{\mu}$ does not depend on x. Moreover,

$$\langle \gamma^{-1}\eta, \gamma^{-1}\xi \rangle_x = \langle \eta, \xi \rangle_{\gamma x},$$

which combined with the previous formulas shows that $\bar{\mu}$ is Γ-invariant. $\square$

We now describe a general procedure to construct a measure on the unit tangent bundle $T^1\tilde{M}$ from the measure $\bar{\mu}$. For every $(\eta, \xi) \in E(\tilde{M})$, the set $P^{-1}(\eta, \xi)$ is a nonempty g_t-invariant subset of $T^1\tilde{M}$. Assume that there exists a family of Borel measures $\{\nu_{\eta,\xi}\}_{(\eta,\xi)\in E(\tilde{M})}$ on $T^1\tilde{M}$ such that:

1. each $\nu_{\eta,\xi}$ is nonzero and supported on $P^{-1}(\eta, \xi)$,
2. for every Borel subset A of $T^1\tilde{M}$, the map
$$(\eta, \xi) \mapsto \nu_{\eta,\xi}(A)$$
is Borel measurable,
3. if $B \subset T^1\tilde{M}$ is bounded, then $\nu_{\eta,\xi}(B)$ is bounded uniformly in $(\eta, \xi) \in E(\tilde{M})$,
4. the family of measures $\{\nu_{\eta,\xi}\}_{(\eta,\xi)\in E(\tilde{M})}$ is Γ-invariant, i.e.
$$\forall \gamma \in \Gamma,\ \gamma_*\nu_{\eta,\xi} = \nu_{\gamma\eta,\gamma\xi}.$$

Then this family of measures $\{\nu_{\eta,\xi}\}_{(\eta,\xi)\in E(\tilde{M})}$ allows us to define a Borel measure $\tilde{\mu}$ on $T^1\tilde{M}$ by writing, for every Borel subset A of $T^1\tilde{M}$,

$$\tilde{\mu}(A) = \int_{E(\tilde{M})} \nu_{\eta,\xi}(A)\, \mathrm{d}\bar{\mu}(\eta, \xi). \tag{3.2}$$

Proposition 3.3.2. *Equation (3.2) defines a locally finite Borel measure $\tilde{\mu}$ on $T^1\tilde{M}$ which is Γ-invariant. This induces a Borel measure μ on T^1M, which locally is the projection of $\tilde{\mu}$. Moreover, if each $\nu_{\eta,\xi}$ is g_t-invariant, then $\tilde{\mu}$ and μ are also g_t-invariant, and if each $\nu_{\eta,\xi}$ is fully supported on the fiber $P^{-1}(\eta, \xi)$, then the supports of $\tilde{\mu}$ and μ are, respectively, $\tilde{\Omega}$ and Ω.*

The choice of the measures $\nu_{\eta,\xi}$ depends slightly on the context. We treat separately the case of nonpositive curvature and the case of compact manifolds without conjugate points.

3.3.1 The Bowen-Margulis measure in nonpositive curvature

For nonpositively curved manifolds, but also for manifolds with no focal points, we proceed as follows. As a consequence of Theorem 2.5.2, for each $(\eta, \xi) \in E(\tilde{M})$, the set $\pi(P^{-1}(\eta, \xi))$ is a totally geodesic flat submanifold of $\tilde{M}$. Then, the geodesic flow g_t acts isometrically on $P^{-1}(\eta, \xi)$. We take the lift of the volume measure of $\pi(P^{-1}(\eta, \xi))$ to $P^{-1}(\eta, \xi)$ as the measure $\nu_{\eta,\xi}$, which is automatically g_t-invariant. In the case that $P^{-1}(\eta, \xi)$ is a single geodesic, $\nu_{\eta,\xi}$ is the Lebesgue measure.

This construction was first carried out by G. Knieper in [Kni98] for the case of nonpositive curvature. His goal was to prove that, under a few hypothesis, μ is the unique measure of maximal entropy for the geodesic flow.

Theorem 3.3.3. *[Kni98] Let M be a nonpositively curved rank 1 compact manifold. Let μ denote the g_t-invariant measure on T^1M constructed in Proposition 3.3.2 from a conformal density $\{\sigma_x\}_{x\in\tilde{M}}$ and the volumes $\{\nu_{\eta,\xi}\}_{(\eta,\xi)\in E(\tilde{M})}$ on the fibers of P. Then, μ is the unique measure of maximal entropy of the geodesic flow g_t on T^1M.*

This result was recently generalized to rank 1 compact manifolds without focal points by F. Liu, F. Wang and W. Wu [LWW20].

Let us now investigate the case of noncompact manifolds. G. Link and J. C. Picaud have studied what happens in nonpositive curvature with few geometric assumptions. They proved a generalization of the Hopf-Tsuji-Sullivan dichotomy (see [Rob03, Théorème 1.7]).

Theorem 3.3.4. *[LP16, Lin18] Let M be a nonpositively curved non-elementary manifold which contains a rank 1 closed geodesic. Let μ denote the g_t-invariant measure on T^1M constructed in Proposition 3.3.2 from a conformal density $\{\sigma_x\}_{x\in\tilde{M}}$ of dimension $\alpha > 0$ and the volumes $\{\nu_{\eta,\xi}\}_{(\eta,\xi)\in E(\tilde{M})}$ on the fibers of P. Then, either:*

1. *The Poincaré series diverges at α, the geodesic flow g_t is ergodic and completely conservative with respect to μ. Moreover,*
   - *(a) the dimension α is equal to the critical exponent δ,*
   - *(b) the conformal density $\{\sigma_x\}_{x\in\tilde{M}}$ is the unique conformal density of dimension δ up to a scalar factor,*
   - *(c) the density $\{\sigma_x\}_{x\in\tilde{M}}$ is ergodic under the action of Γ,*
   - *(d) σ_x has no point masses.*
2. *The Poincaré series converges at α, the geodesic flow g_t is completely dissipative with respect to μ.*

R. Ricks proved independently a weaker version of Theorem 3.3.4 for rank 1 $CAT(0)$ spaces with a different construction of μ [Ric17]. These two constructions coincide when Γ is divergent. We do not know if the previous theorem has been generalized for manifolds without focal points.

We observe that if the measure μ is finite, then g_t is automatically conservative, so we are in the first case of Theorem 3.3.4. Since μ is locally finite, it is finite if M is compact. We also observe that when Γ is divergent, the unique conformal density is constructed explicitly in Proposition 3.2.4.

Corollary 3.3.5. *Let M be a nonpositively curved non-elementary manifold which contains a rank 1 closed geodesic. Assume that Γ is divergent and let $\{\sigma_x\}_{x\in\tilde{M}}$ be the unique conformal density of dimension δ. Then the rank 1 set has full measure in T^1M. In particular, in restriction to R_1, the measure $\tilde{\mu}$ satisfies*

$$\mathrm{d}\tilde{\mu}(v) = e^{2\delta\langle v_-,v_+\rangle_0}\mathrm{d}t\,\mathrm{d}\sigma_0(v_-)\,\mathrm{d}\sigma_0(v_+). \tag{3.3}$$

Proof. Let $v \in T^1M$ be a rank 1 vector tangent to a closed geodesic. Then v is nonwandering, so it is in the support of μ. Since the rank 1 set is open, it has positive measure. This set is also g_t-invariant, so we conclude thanks to the ergodicity that R_1 has full measure. □

3.3.2 The Bowen-Margulis measure on compact manifolds without conjugate points

Outside the no focal points case, since the flat strip theorem may fail [Bur92], the volume measure on the fibers of P is not necessarily g_t-invariant. V. Climenhaga,

G. Knieper and K. War succeeded in defining a g_t -invariant measure on T^1M for compact visibility manifolds and deduce relevant dynamical properties. They use the following fact. Recall that the set $\tilde{I}(v)$ is defined as the intersection of the stable and the unstable horosphere of v.

Lemma 3.3.6. *Let M be a compact visibility manifold. There is a Γ-invariant measurable map $F : \{\tilde{I}(v) \,|\, v \in T^1\tilde{M}\} \to T^1\tilde{M}$ such that $F(\tilde{I}(v)) \in \tilde{I}(v)$ for all $v \in T^1\tilde{M}$.*

Recall that, for a visibility manifold, any two distinct points at infinity are joined by a geodesic, so $E(\tilde{M}) = \partial^2\tilde{M}$. Next, we can define the measures $\nu_{\eta,\xi}$. For $(\eta,\xi) \in \partial^2\tilde{M}$, choose any $v \in P^{-1}(\eta,\xi)$ and, for a Borel subset A of $T^1\tilde{M}$, put

$$\nu_{\eta,\xi}(A) = \mathrm{Leb}\{t \in \mathbb{R} \mid F(\tilde{I}(g_t v)) \in A\}.$$

This definition does not depend on v. Hence we have a family of measures $\{\nu_{\eta,\xi}\}_{(\eta,\xi)\in\partial^2\tilde{M}}$ satisfying the conditions of Proposition 3.3.2, so we obtain a finite measure μ on T^1M. This measure is not necessarily g_t-invariant yet, for this reason we consider a weak limit of

$$\frac{1}{T}\int_0^T g_{t*}\mu \mathrm{d}t$$

when $T \to +\infty$. This gives a g_t-invariant measure μ_0 on T^1M that we can assume to be normalized.

We denote by $\mathrm{Prob}(g_1)$ the set of Borel g_1-invariant probability measures. For $\nu \in \mathrm{Prob}(g_1)$, we write $\tilde{\nu}$ for the lift of ν to $T^1\tilde{M}$. We also denote the metric entropy of ν by $h_\nu(g_1)$ and the topological entropy of the geodesic flow by h_{top}.

Theorem 3.3.7. *Let M be a compact manifold without conjugate points and let $\{\sigma_x\}_{x\in\tilde{M}}$ be the conformal density given by Proposition 3.2.4. We assume that:*

1. *There exists a metric g' on M of negative curvature.*
2. *Geodesic rays on $\tilde{M}$ diverge.*
3. *The covering group Γ is residually finite (the intersection of its finite index subgroups is trivial).*
4. *M has an entropy gap:*

$$\sup\{h_\nu(g_1) \,|\, \nu \in \mathrm{Prob}(g_t), \tilde{\nu}(\mathcal{E}) = 0\} < h_{top}.$$

Then μ_0 is the unique measure of maximal entropy. Moreover, μ_0 is ergodic, fully supported on T^1M and its lift $\tilde{\mu}_0$ gives full measure to the expansive set $\mathcal{E}$.

The ergodicity of μ_0 is a consequence of the fact that μ_0 is the unique measure of maximal entropy. One particular case of manifolds satisfying the hypothesis of the theorem are compact surfaces without conjugate points of higher genus (see Proof of Theorem 1.1 in [CKW21]).

Corollary 3.3.8. *Let M be a compact surface without conjugate points of genus equal or higher than 2. Then μ_0 is the unique measure of maximal entropy. Moreover, μ_0 is ergodic, fully supported on T^1M and its lift $\tilde{\mu}_0$ gives full measure to the expansive set $\mathcal{E}$.*

Since the restriction of μ to $\mathcal{E}$ is already g_t-invariant, we get the following corollary as well.

Corollary 3.3.9. *Let M be a compact manifold satisfying the hypothesis of Theorem 3.3.7. For $\bar{\mu}$-a.e. $(\eta,\xi) \in \partial^2\tilde{M}$, $P^{-1}(\eta,\xi)$ is a single orbit of g_t. In particular, the measures μ and μ_0 coincide and we can write, in restriction to $\mathcal{E}$,*

$$\mathrm{d}\tilde{\mu}_0(v) = e^{\delta\langle v_-,v_+\rangle_0}\mathrm{d}t\,\mathrm{d}\sigma_0(v_-)\,\mathrm{d}\sigma_0(v_+). \tag{3.4}$$

Corollaries 3.3.5 and 3.3.9 give the same description of the measure μ for manifolds satisfying simultaneously their hypothesis. Based on the terminology for Anosov flows, and in particular geodesic flows in negative curvature, the measure μ will be called the Bowen-Margulis measure.

Definition 3.3.3. We refer to both the unique measure μ of Theorem 3.3.4 when $\tilde{M}$ is a nonpositively curved rank 1 manifold whose covering group Γ is divergent and the measure μ_0 of Theorem 3.3.7 when M is a compact manifold satisfying the hypothesis therein as the Bowen-Margulis measure. From now on, we denote these measures by μ_{BM}.

3.4 A family of measures on the horospheres

We now define a natural family of measures on the horospheres from a Patterson-Sullivan measure. In the whole section, M denotes either a nonpositively curved rank 1 non-elementary manifold or a compact visibility manifold without conjugate points.

On a stable horosphere $\tilde{H}^s(v), v \in T^1\tilde{M}$, we define a projection map

$$P_{\tilde{H}^s(v)} : \tilde{H}^s(v) \to \partial\tilde{M} \setminus \{v_+\},$$

which takes a vector $w \in \tilde{H}^s(v)$ to its negative endpoint $w_- \in \partial\tilde{M}$. Similarly, for an unstable horosphere $\tilde{H}^u(v)$, there is a map $P_{\tilde{H}^u(v)} : \tilde{H}^u(v) \to \partial\tilde{M} \setminus \{v_-\}$, sending $w \in \tilde{H}^u(v)$ to w_+. These two maps are continuous. Nonempty fibers of $P_{\tilde{H}^s(v)}$ and $P_{\tilde{H}^u(v)}$ are sets of the form $\tilde{I}(w)$.

Recall that, if M is nonpositively curved, $\pi(\tilde{I}(v))$ is a flat totally geodesic submanifold of $\tilde{M}$ for every $v \in T^1\tilde{M}$, possibly a single point. When M is a compact visibility manifold without conjugate points, Lemma 3.3.6 provides a measurable Γ-invariant map F, which for each $\tilde{I}(v)$ selects a vector $F(\tilde{I}(v)) \in \tilde{I}(v)$.

We consider a family of measures $\{\nu_{\tilde{I}(v)}\}_{v\in T^1\tilde{M}}$ supported on the sets $\tilde{I}(v)$. The measure $\nu_{\tilde{I}(v)}$ is set to be either

1. the lift to $\tilde{I}(v)$ of the volume measure of $\pi(\tilde{I}(v))$ if M is nonpositively curved, or

2. the Dirac measure at $F(\tilde{I}(v))$ if M is a compact visibility manifold.

This family of measures are Γ-invariant and depend measurably on $\tilde{I}(v)$. When $\tilde{I}(v)$ is a single vector, $\nu_{\tilde{I}(v)}$ is the Dirac measure on this vector. If we integrate these measures along $\tilde{S}(v) = \cup_{t\in\mathbb{R}}\tilde{I}(g_t v)$ in t, we recover the measure ν_{v_-,v_+} supported on $\tilde{S}(v) = P^{-1}(v_-,v_+)$.

Let H be a stable or unstable horosphere on $T^1\tilde{M}$. For $\xi \in P_H(H)$, the function $\beta_\xi(0, \cdot\,)$ is constant on $\pi(P_H^{-1}(\xi))$. So the quantity

$$\phi_H(\xi) := e^{\delta\beta_\xi(0,\pi(P_H^{-1}(\xi)))}$$

is well defined.

Definition 3.4.1. Let H be a stable or unstable horosphere on $T^1\tilde{M}$. For every Borel subset A of H, we define

$$\mu_H(A) = \int_{P_H(H)} \nu_{P_H^{-1}(\xi)}(A)\phi_H(\xi)\,\mathrm{d}\sigma_0(\xi). \tag{3.5}$$

It is straightforward to prove the following properties.

Proposition 3.4.1. *Equation (3.5) defines a family* $\{\mu_{\tilde{H}^u(v)}\}_{v\in T^1\tilde{M}}$ *of locally finite Borel measures on the unstable horospheres of* $T^1\tilde{M}$ *such that*

1. *they are* Γ*-invariant, i.e.*

$$\gamma_*\mu_{\tilde{H}^u(v)} = \mu_{\tilde{H}^u(\gamma v)},$$

2. *they are uniformly expanded by the geodesic flow, i.e.*

$$\mu_{g_t\tilde{H}^u(v)} = e^{\delta t}(g_t)_*\mu_{\tilde{H}^u(v)}.$$

There is a family of measures $\{\mu_{\tilde{H}^u(v)}\}_{v\in T^1\tilde{M}}$ *on the stable horospheres, which is uniformly contracting instead of expanding. The definitions do not depend on the point* $0 \in \tilde{M}$.

The measures on the horospheres that we have defined are analogous to the Margulis measures on the stable and the unstable manifolds of an Anosov flow. In the remaining chapters, they will be used to prove ergodic properties of the horospheres.

Chapter 4

Equidistribution of horospheres under the action of the geodesic flow

In this entire section M will be a non-elementary rank 1 nonpositively curved manifold. For our purpose, we assume that the Bowen-Margulis measure μ_{BM} on the space T^1M is finite, hence the geodesic flow is conservative, according to the Poincaré recurrence theorem, and there is only one conformal density σ_0. Our goal is to find an equidistribution result in the sense that the μ_H-averages of functions on a horosphere H tend to the μ_{BM}-averages on the whole space.

As we mentioned before, there are connections between the dynamics of the geodesic flow and of the horospheres. The equidistribution result that we want to prove relies on the mixing property of the geodesic flow with respect to the Bowen-Margulis measure. The first section guarantees this property for a large class of nonpositively curved rank 1 manifolds. It is a generalization of a result already known for negatively curved manifolds ([Dal00, Theorem A] and [Bab02, Theorem 1]). In Section 2, we will prove the main theorem of this chapter about the equidistribution of horospheres under the action of the geodesic flow. This chapter corresponds to the first part of [BC21a].

4.1 Topological mixing of the geodesic flow

The starting point is the mixing property of the geodesic flow with respect to the Bowen-Margulis measure μ_{BM}. The next result says that this property is equivalent to the topological mixing of the geodesic flow on Ω. We do not know if this equivalence has been stated in this generality, although it can be expected and the main part of the work is already published.

There is a third equivalent property related to the length of the closed geodesics, analogous to what happens in negative curvature. We define the *rank* 1 *length spectrum* as the set of lengths of rank 1 closed geodesics. We say that the rank 1 length spectrum is *non-arithmetic* if the rank 1 length spectrum generates a dense subgroup of $\mathbb{R}$.

Theorem 4.1.1. *Let M be a rank* 1 *nonpositively curved non-elementary complete connected Riemannian manifold. Assume that the Bowen-Margulis measure μ_{BM} is finite. Then the following are equivalent:*

(i) The geodesic flow g_t is topologically mixing on the nonwandering set Ω.

(ii) The geodesic flow g_t is mixing with respect to the Bowen-Margulis measure μ_{BM}.

(iii) The rank 1 length spectrum is non-arithmetic.

Proof. (ii) $\implies$ (i) The mixing property with respect to a measure implies the topological mixing on the support of the measure. In our case, the support of μ_{BM} is the nonwandering set Ω, so the implication is proved.

(i) $\implies$ (iii) We reproduce the argument used in negative curvature by F. Dal'bo [Dal00]. Since the set of rank 1 vectors is open [Bal95], we can find a closed ball B of certain radius only containing rank 1 vectors. Let $\varepsilon > 0$ be a given number. We apply the closing lemma for the nonwandering rank 1 set proved by Y. Coudène and B. Schapira [CS10]: there exists constants $T_0 > 0$ and $\delta > 0$ such that, for every $v \in B$ and $t \geq T_0$ with $d_1(v, g_t(v)) \leq \delta$, there exists a periodic rank 1 vector v' at distance $d_1(v, v') \leq \varepsilon$, where the period t' of v satisfies $|t - t'| < \varepsilon$.

There exists a nonempty open subset U of Ω of diameter smaller than δ and such that $U \subset B$. Since the geodesic flow on Ω is topologically mixing, there exists a number $T \geq T_0$ such that for all $t \geq T$, we have $U \cap g_t(U) \neq \emptyset$. In particular, there is a rank 1 vector v in B satisfying $d_1(v, g_t(v)) \leq \delta$. Hence, for each $t \geq T$, there exists a periodic rank 1 vector of period in $[t - \varepsilon, t + \varepsilon]$. Since ε is arbitrary, this proves that the rank 1 length spectrum is non-arithmetic.

(iii) $\implies$ (ii) This implication may be the hardest, but it is essentially done in the proof of Theorem 2 in [Bab02], asserting that the geodesic flow is mixing with respect to μ_{BM} on a compact manifold. All the arguments work for a rank 1 manifold with finite Bowen-Margulis measure, but at the end, instead of applying the compactness, we can use the assumption of non-arithmeticity of the length spectrum. $\square$

4.2 Equidistribution of horospheres

We start with a local result showing that there is equidistribution near rank 1 vectors: for a function $f : T^1M \to \mathbb{R}$, the average on a horosphere of its lift $\tilde{f} : T^1\tilde{M} \to \mathbb{R}$ pushed by the geodesic flow converges to the average of f with respect to the Bowen-Margulis measure.

Proposition 4.2.1. *Let M be a nonpositively curved non-elementary complete connected Riemannian manifold with a closed rank 1 geodesic. Assume that the geodesic flow g_t on T^1M is topologically mixing on Ω and that the Bowen-Margulis measure μ_{BM} is finite. Then, for every rank 1 vector $v \in \tilde{\Omega} \subset T^1\tilde{M}$, there exists an open subset U of $\tilde{H}^u(v)$ containing v which is equidistributed under the action of the geodesic flow; i.e. for every bounded and uniformly continuous function f on T^1M and every Borel neighborhood $V \subset U$ of v we have*

$$\frac{1}{\mu_{\tilde{H}^u(v)}(V)} \int_V \tilde{f} \circ g_t \, \mathrm{d}\mu_{\tilde{H}^u(v)} \xrightarrow[t \to +\infty]{} \frac{1}{\mu_{BM}(T^1M)} \int_{T^1M} f \, \mathrm{d}\mu_{BM}.$$

Proof. We follow the same strategy as M. Babillot in [Bab02], which consists in approximating the integral on a piece of horosphere by the integral of the same function on a box around that piece, and then use the mixing property of the geodesic flow with respect to μ_{BM}. The added difficult is to find a box with a good system of coordinates, which is done by avoiding the higher rank vectors.

Let v be a rank 1 vector in $\tilde{\Omega}$ and denote its horosphere by H. From [Bal95, Lemma III.3.1] we know that there exist disjoint connected neighborhoods A_1 and A_2 of v_- and v_+, respectively, in $\partial\tilde{M}$ such that for every $(\xi, \eta) \in A_1 \times A_2$ there exists a unique geodesic from ξ to η, and it has rank 1. This allows us to consider a coordinate neighborhood of v via the map $\bar{P}$ of the form $A_1 \times A_2 \times \mathbb{R}$.

We claim that the proposition is true with $U = P_H^{-1}(A_2)$. Notice that U is contained in the rank 1 set by construction. Consider any neighborhood $V \subset U$ of v and write $V_+ := \{w_+ \,|\, w \in V\}$ for its projection to the boundary at infinity $\partial\tilde{M}$. Since v is nonwandering, its endpoints are in the limit set, and this guarantees that V_+ and V have positive measure. We notice that the integral on V of a function h of $T^1\tilde{M}$ can be written in coordinates as

$$\int_V h \,\mathrm{d}\mu_{\tilde{H}^u(v)} = \int_{V_+} h(v_-, \eta, t_0) e^{\delta\beta_\eta(0,\pi(v_-,\eta,t_0))} \mathrm{d}\sigma_0(\eta),$$

where t_0 has the value $\beta_{v_-}(0, \pi(v))$. This is because $\nu_{P_H^{-1}(\eta)}$ is the Dirac measure on $P_H^{-1}(\eta)$ whenever this set is contained in the rank 1 set.

Given $\varepsilon > 0$, we can find a small connected neighborhood $B \subset A_1$ of v_- and a number $r > 0$ such that:

(i) $\forall \xi \in B,\ \forall \eta \in V_+, \quad 1 - \varepsilon \leq e^{\delta\beta_\eta(\pi(v_-,\eta,t_0),\, \pi(\xi,\eta,t_0))} \leq 1 + \varepsilon,$

(ii) $\forall (\xi, \eta) \in B \times V_+, \forall s \in [-r, r], \forall t \geq 0, \quad |\tilde{f}(v_-, \eta, t_0 + t) - \tilde{f}(\xi, \eta, t_0 + t + s)| < \varepsilon.$

The first property follows from the continuity of the map $\bar{P}$ on the coordinate neighborhood, and the continuity of the projection π and of the Busemann function. We use that V_+ is relatively compact to assert that the inequality holds uniformly in $\eta \in V_+$. For property (ii), we apply the uniform continuity of $\tilde{f}$, and then we choose B and r so that the points (v_-, η, t_0) and $(\xi, \eta, t_0 + s)$ are close enough for $\xi \in B$ and $s \in [-r, r]$, uniformly in $\eta \in V_+$. Since these points are in the same weak stable leaf, the distance between them does not increase when they are pushed by the geodesic flow, which allows us to deduce the above property for all $t \geq 0$. Again the condition $v \in \tilde{\Omega}$ implies that $v_- \in \Lambda$, which ensures that B has positive measure.

These estimates allow one to compare the average of $\tilde{f} \circ g_t$ on the set V with respect to $\mu_{\tilde{H}^u(v)}$ and the average of the same function on the box of the form $\bar{P}^{-1}(B \times V_+ \times [t_0, t_0 + r])$ with respect to the measure μ_{BM} by means of the product structure of μ_{BM} in the rank 1 set (Equation (3.3)). More precisely, for all nonnegative t,

$$\left[\frac{\int_V \tilde{f} \circ g_t \,\mathrm{d}\mu_{\tilde{H}^u(v)}}{\mu_{\tilde{H}^u(v)}(V)} - \varepsilon\right] \frac{1 - \varepsilon}{1 + \varepsilon} \leq \frac{\int_{\bar{P}^{-1}(B \times V_+ \times [t_0, t_0 + r])} \tilde{f} \circ g_t \,\mathrm{d}\mu_{BM}}{\mu_{BM}(\bar{P}^{-1}(B \times V_+ \times [t_0, t_0 + r]))} \leq$$

$$\left[\frac{\int_V \tilde{f} \circ g_t \,\mathrm{d}\mu_{\tilde{H}^u(v)}}{\mu_{\tilde{H}^u(v)}(V)} + \varepsilon\right] \frac{1 + \varepsilon}{1 - \varepsilon}.$$

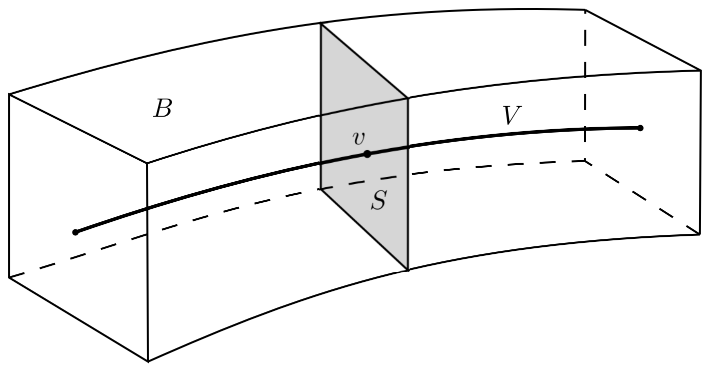

Figure 4.1: We consider a box B around a subset V of an unstable horosphere. A vertical leaf S of B is included in the weak stable manifold, so its size does not increase when we apply the geodesic flow for a positive time. Hence, if the box B is thin enough, $\tilde{f} \circ g_t(v)$ is a good approximation of $\tilde{f} \circ g_t$ on S.

See Figure 4.1 for a schematic representation of these approximations.

Moreover we may assume that the neighborhood $\bar{P}^{-1}(B \times A_2 \times [t_0, t_0 + r]) \subset T^1\tilde{M}$ is homeomorphic to its projection on the unit tangent bundle of the manifold M. Then, since the geodesic flow is mixing with respect to μ_{BM}, the average of $\tilde{f} \circ g_t$ in $\bar{P}^{-1}(B \times V_+ \times [t_0, t_0 + r])$ converges to $\frac{1}{\mu_{BM}(T^1M)} \int f \mathrm{d}\mu_{BM}$ when t goes to infinity. We have thus shown the equidistribution of U.

□

To deduce a global result, we need to understand what happens on vectors of rank different from 1, and the next two lemmas will be crucial. The *unstable manifold* of v in $T^1\tilde{M}$ is the set

$$\tilde{W}^u(v) = \{w \in T^1\tilde{M} \mid d_1(g_t(v), g_t(w)) \to 0,\, t \to -\infty\}.$$

$\tilde{W}^u(v)$ is a subset of the unstable horosphere $\tilde{H}^u(v)$, but they are not necessarily equal in nonpositive curvature.

Lemma 4.2.2. *Let M be a rank 1 nonpositively curved non-elementary complete connected Riemannian manifold. If v is a rank 1 recurrent vector in $T^1\tilde{M}$, then its unstable horosphere coincides with its unstable manifold, $\tilde{H}^u(v) = \tilde{W}^u(v)$, and it consists of rank 1 vectors exclusively.*

Proof. The fact that the unstable manifold and the horosphere coincide is already proved in [Kni98, Proposition 4.1]. Let w in $\tilde{W}^u(v)$ and r its rank, we will see that r is 1. Since v is negatively recurrent there exist a sequence $t_n \to -\infty$ and isometries $\gamma_n \in \Gamma$ such that $\gamma_n(g_{t_n}(v)) \to v$ when $n \to \infty$. Now we have

$$d_1(v, \gamma_n g_{t_n}(w)) \leq d_1(v, \gamma_n g_{t_n}(v)) + d_1(g_{t_n}(v), g_{t_n}(w)) \longrightarrow 0$$

and the rank of $\gamma_n g_{t_n}(w)$ is the same as the rank of w, r. Since v is a limit of vectors of rank r and the rank function is upper semi-continuous, we deduce $r \leq \operatorname{rank} v = 1$. □

Lemma 4.2.3. *Let M be a nonpositively curved non-elementary complete connected Riemannian manifold with a closed rank 1 geodesic. Assume that the Bowen-Margulis measure μ_{BM} is finite and that the geodesic flow g_t on T^1M is* *ergodic with respect to the μ_{BM}. Then, for every horocycle H, the set of vectors in H of rank equal or higher than 2 is μ_H-negligible.*

Proof. Let $\tilde{Rec}^1 \subset T^1\tilde{M}$ be the set of rank 1 vectors which are recurrent under g_t on the quotient T^1M and $S \subset T^1\tilde{M}$ be the set of vectors of rank 2 or higher. We claim that the projections to the boundary of these two sets are disjoint, $\tilde{Rec}^1_+ \cap S_+ = \emptyset$. Otherwise, there are vectors $v \in \tilde{Rec}^1$ and $w \in S$ such that $v_+ = w_+$. By Lemma 4.2.2, the unstable horosphere of $-v$ only contains vectors of rank 1. The geodesic associated to $-w$ intersects this horosphere $\tilde{H}^u(-v)$ (Figure 4.2), so w should have rank 1, which is a contradiction.

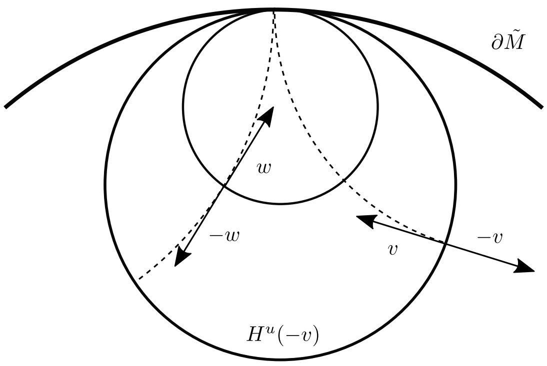

Figure 4.2: Vectors v and w of the proof.

Around a nonwandering rank 1 vector there is a neighborhood only consisting of rank 1 vectors, and this neighborhood has positive measure because it intersects the support of μ_{BM}. By hypothesis, the manifold M contains a closed rank 1 geodesic, which is an example of nonwandering rank 1 geodesic. The set of rank 1 vectors has positive measure, and it is invariant under the geodesic flow. So the set of rank 1 vectors has full measure because of the ergodicity of μ_{BM}. In consequence, the set of rank 1 recurrent vectors Rec^1 has also full μ_{BM}-measure in view of the Poincaré recurrence theorem. By the product structure of μ_{BM}, we see that $\tilde{Rec}^1_+$ has positive σ_0-measure. Finally, $\tilde{Rec}^1_+$ is a Γ-invariant set, so we deduce that $\tilde{Rec}^1_+$ has full σ_0-measure because Γ acts ergodically.

Therefore, S_+ is negligible. The endpoints of higher rank vectors in $\tilde{H}^u(v)$ are clearly in S_+ and, using the definition of the measure on the horosphere, we obtain $\mu_{\tilde{H}^u(v)}(S \cap \tilde{H}^u(v)) = 0$. □

We can finally prove Theorem A, which we have reformulated in terms of horospheres on the universal cover $\tilde{M}$. On the horospheres centered at the limit set, every open set with positive and finite measure is equidistributed (Figure 4.3). Being positive is equivalent to having a nonwandering rank 1 vector. In particular, all relatively compact neighborhoods of nonwandering rank 1 vectors are equidistributed.

Theorem 4.2.4. *Let M be a nonpositively curved non-elementary complete connected Riemannian manifold with a closed rank 1 geodesic. Assume that the geodesic flow g_t on T^1M is topologically mixing on Ω and that the Bowen-Margulis measure μ_{BM} is finite. Then, for every horosphere $H \subset T^1\tilde{M}$ centered at Λ, every open subset U of H of finite and nonzero μ_H-measure is equidistributed under*

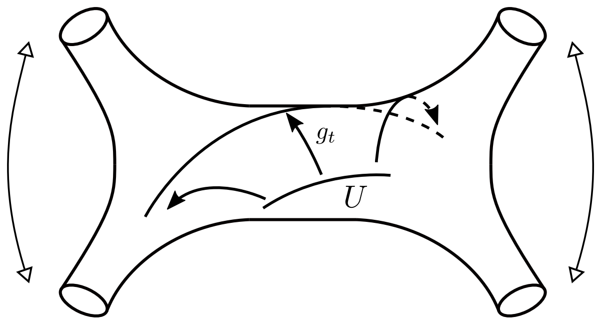

Figure 4.3: The average of f on the image of an open subset U of a horosphere H by the geodesic flow g_t with respect to μ_H tends to the average of f with respect to μ_{BM}.

the action of the geodesic flow; i.e. for every bounded and uniformly continuous function f on T^1M, we have

$$\frac{1}{\mu_H(U)}\int_U \tilde{f}\circ g_t \,\mathrm{d}\mu_H \xrightarrow[t\to+\infty]{} \frac{1}{\mu_{BM}(T^1M)}\int_{T^1M} f\,\mathrm{d}\mu_{BM}.$$

Proof. We first observe that the set U^1 of rank 1 vectors in U is open in H, because the set of rank 1 vectors is open in $T^1\tilde{M}$ [Bal95]. By Lemma 4.2.3, the set U^1 has full measure in U, so the averages on the two sets are the same. Next, we use the fact that μ_H is a Radon measure: given a number $\varepsilon > 0$, there exists a compact subset $K \subset U^1$ such that $\mu_H(U^1 \setminus K) < \varepsilon$.

Since $\tilde{\Omega}$ is closed, $L = K \cap \tilde{\Omega}$ is again compact, and L has full measure in K, because vectors outside of $\tilde{\Omega}$ are not in the support of μ_H. We want to show that L is equidistributed. Proposition 4.2.1 gives an equidistributed open neighborhood $U_v \subset U^1$ of each vector v in L. The set L can be covered by finitely many U_v because it is compact, say $U_1, \ldots, U_m$. Then, the sets $V_k = U_k \setminus \cup_{i=1}^{k-1} U_i$ form a finite cover of L by pairwise disjoint Borel sets, and each of them is equidistributed, thanks to the fact that the subsets of U_v are equidistributed too.

If we let $\lambda := \int f \mathrm{d}\mu_{BM}/\mu_{BM}(T^1M)$, the set $V := V_1 \cup \cdots \cup V_n$ is equidistributed because

$$\frac{\int_V \tilde{f}\circ g_t \,\mathrm{d}\mu_H}{\mu_H(V)} = \frac{\sum_{i=1}^n \int_{V_i} \tilde{f}\circ g_t \,\mathrm{d}\mu_H}{\mu_H(V)} \xrightarrow[t\to+\infty]{} \frac{\sum_{i=1}^n \mu_H(V_i)\lambda}{\mu_H(V)} = \lambda.$$

On the other hand, we have $\mu_H(U \setminus V) < \varepsilon$, so

$$\left|\frac{1}{\mu_H(U)}\int_U \tilde{f}\circ g_t \,\mathrm{d}\mu_H - \frac{1}{\mu_H(V)}\int_V \tilde{f}\circ g_t \,\mathrm{d}\mu_H\right| \le \frac{2\varepsilon \,\|f\|_\infty}{\mu_H(U)}$$

for all $t \geq 0$. This proves that U is equidistributed as well. □

Chapter 5

Horocyclic flows in nonpositive curvature

In this chapter we restrict our attention to rank 1 surfaces with nonpositive curvature. Our goal is to study the invariant measures of the horocyclic flow aiming for unique ergodicity results. To prove some of these results we will make use of the equidistribution theorem of Chapter 4.

The first step is to define a flow that preserves the Bowen-Margulis measure and whose orbits are horocycles. The idea is to define the parametrization of the flow by the measures on the horocycles as in the negative curvature case [Mar75b]. However, the presence of flat pieces of horocycles makes impossible to define globally a continuous flow with this method. We define a subset Σ_0 of the unit tangent bundle which excludes the horocycles causing trouble, like the one displayed in Figure 5.2, and which is topologically and metrically large. We will define a parametrization of the horocyclic flow on Σ_0 and prove that it is uniquely ergodic. This is done in Section 5.1.

In Section 5.2, we study the other parametrizations of the horocyclic flow, and in particular the parametrization by arc-length. The goal is to show that unique ergodicity is still true for these other parametrizations. This is clear when we have two continuous flows with no fixed points and with the same orbits on a compact space, but here the subset Σ_0 where the Margulis parametrization is defined is not compact. The key point is to show that the change of time between the two flows does not blow up outside Σ_0.

Finally, in Section 5.3, we obtain a stronger result for the class of compact nonpositively curved surfaces without flat strips. In this case, we will see that the Margulis parametrization is defined everywhere and not only on Σ_0. Using this parametrization, we will show that the horocyclic flow is uniquely ergodic on the whole unit tangent bundle.

The results in this chapter correspond to the second part of [BC21a] and to [BC21b]. These were the first attempts made by the author to solve the problem of unique ergodicity, and for this reason they are not conclusive. Chapter 6 provides a complete answer in the compact case with slightly different methods.

5.1 Margulis horocyclic flow

5.1.1 Surfaces with nonpositive curvature

In this section, M is a nonpositively curved non-elementary orientable surface with a closed rank 1 geodesic and the Bowen-Margulis measure μ_{BM}, constructed as before, is assumed to be finite. We will further assume that M satisfies the duality condition, which means that every vector of T^1M is nonwandering, or equivalently we assume that $\Lambda = \partial\tilde{M}$. Under these hypothesis, the geodesic flow is topologically mixing [Ebe73a, Theorem 6.3], so it is also mixing with respect to the Bowen-Margulis measure. The duality condition is satisfied if M has finite Riemannian volume, as an application of the Poincaré recurrence theorem. It is worth mentioning that a nonpositively curved non-elementary rank 1 manifold satisfying the duality condition contains automatically a closed rank 1 geodesic.

Moreover, we know that any two distinct points in the boundary at infinity can be connected by a geodesic. This follows from the fact that, for a nonflat surface M with the duality condition, the universal cover $\tilde{M}$ satisfies the visibility axiom [Ebe79, Proposition 2.5]. Therefore, the map $P : T^1\tilde{M} \to \partial^2\tilde{M}$ is surjective.

We notice that an orientation of the boundary at infinity $\partial\tilde{M}$ induces an orientation on each horocycle of $T^1\tilde{M}$. One vector $v \in T^1\tilde{M}$ divides its horocycle $\tilde{H}^u(v)$ in two connected sets, one in the positively oriented direction, $\tilde{H}^u_R(v)$, and the other in the negatively oriented direction, $\tilde{H}^u_L(v)$. The group of isometries Γ is orientation-preserving because M is orientable. As a consequence, horocycles on $T^1\tilde{M}$ descend to T^1M as oriented immersed curves.

Horocycles are diffeomorphic to the real line. Let H be a horocycle of $T^1\tilde{M}$. The interval $(v, w) \subset H$ between two vectors $v, w \in H$ is the connected subset bounded by v and w. The map $P_H : H \to \partial\tilde{M} \setminus \{\xi\}$, where ξ is the center of H, which projects a vector to its positive endpoint, is continuous and surjective. We also observe that $P_H(v) = P_H(w)$, with $v \neq w$, implies, according to the flat strip theorem, that the curvature vanishes on the strip $\pi(\cup_{t\in\mathbb{R}} g_t((v, w)))$. Such an interval (v, w) will be called a *flat piece of horocycle* (see Figure 5.1). It is clear that H does not contain any flat piece if and only if P_H is injective, in which case P_H is also a homeomorphism.

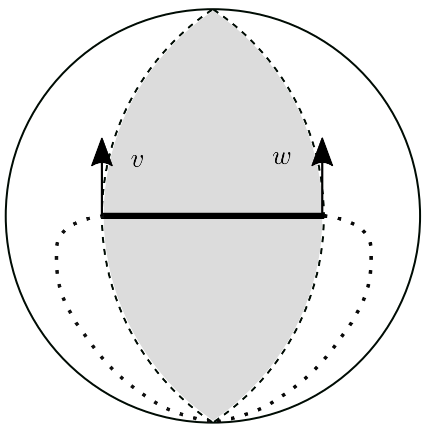

Figure 5.1: Universal cover of the surface M with a region where the curvature vanishes (shadowed region). We represent an unstable horocycle with a flat piece.

5.1.2 Continuity of the measures on the horocycles

Let M be a connected complete non-elementary Riemannian surface of nonpositive curvature with a closed rank 1 geodesic. We assume additionally that the group Γ is divergent. The next result expresses the continuity of the measure μ_H with respect to the horocycle H, and is proved from the simplified expressions of the measures.

Proposition 5.1.1. *The map*

$$\begin{array}{ccc} \{(v,w) \in T^1\tilde{M} \times T^1\tilde{M} \mid w \in \tilde{H}^u(v)\} & \longrightarrow & \mathbb{R} \\ (v,w) & \longmapsto & \mu_{\tilde{H}^u(v)}((v,w)) \end{array}$$

is continuous.

Proof. The measure of the interval (v,w) is

$$\mu_{\tilde{H}^u(v)}((v,w)) = \int_{P_{\tilde{H}^u(v)}((v,w))\setminus\tilde{S}_+} \mathbf{1}_{(v,w)}(P^{-1}_{\tilde{H}^u(v)}(\eta))\phi_{\tilde{H}^u(v)}(\eta)\,\mathrm{d}\sigma_0(\eta).$$

The set $P_{\tilde{H}^u(v)}((v,w))$ is an interval of $\partial\tilde{M}$ that satisfies $(v_+,w_+) \subset P_{\tilde{H}^u(v)}((v,w)) \subset [v_+,w_+]$. Since σ_0 has no point masses [LP16, Proposition 5], we can write

$$\mu_{\tilde{H}^u(v)}((v,w)) = \int_{(v_+,w_+)\setminus\tilde{S}_+} \phi_{\tilde{H}^u(v)}\,\mathrm{d}\sigma_0.$$

Let $f : T^1\tilde{M} \to \mathbb{R}$ be the real continuous function given by

$$f(u) = \exp(\delta\beta_{u_+}(0,\pi(u))).$$

Because of the definition of $\phi_{\tilde{H}^u(v)}$ and the fact that $(v_+,w_+)\setminus\tilde{S}_+ \subset P_{\tilde{H}^u(v)}(\tilde{H}^u(v)\cap\tilde{R}_1)$, we have

$$\forall\eta \in (v_+,w_+)\setminus\tilde{S}_+, \quad \phi_{\tilde{H}^u(v)}(\eta) = e^{\delta\beta_\eta(0,\pi(P^{-1}_{\tilde{H}^u(v)}(\eta)))} = f(P^{-1}_{\tilde{H}^u(v)}(\eta)).$$

Let $v \in T^1\tilde{M}$ and $w \in \tilde{H}^u(v)$. We want to show the continuity of the map in the statement at (v,w). Let us explain step by step the needed estimates and at the end we will apply them. We fix $\varepsilon > 0$.

Boundedness of the integrated function. Let K be a compact neighborhood of $[v,w]$ in $T^1\tilde{M}$. By the continuity of horocycles there are open neighborhoods $V_1 \subset K$ and $W_1 \subset K$ of v and w in $T^1\tilde{M}$ such that if $v' \in V_1$ and $w' \in W_1\cap\tilde{H}^u(v')$ the interval (v',w') is contained in K. The function f is bounded on K by a constant $C > 0$.

For all $v' \in V_1$, for all $w' \in W_1 \cap \tilde{H}^u(v')$, we have the inclusion

$$P^{-1}_{\tilde{H}^u(v')}((v'_+,w'_+)\setminus\tilde{S}_+) \subset (v',w') \subset K,$$

which says that, for all $\eta \in (v'_+,w'_+)\setminus\tilde{S}_+$, the quantity $f(P^{-1}_{\tilde{H}^u(v')}(\eta))$ is bounded by C.

Approximation of intervals. Since σ_0 has no point masses and is outer regular, there are open intervals A and B around v_+ and w_+ in $\partial\tilde{M}$ of arbitrarily small measure. We choose A and B such that $\sigma_0(A) < \varepsilon/16C$ and $\sigma_0(B) < \varepsilon/16C$. By the continuity of the projection $T^1\tilde{M} \to \partial\tilde{M}$ to the endpoint, there are neighborhoods V_2 and W_2 of v and w such that $V_{2+} \subset A$ and $W_{2+} \subset B$. For every $v' \in V_2$ and $w' \in W_2$, we have

$$(v_+, w_+)\triangle(v'_+, w'_+) \subset [v_+, v'_+] \cup [w_+, w'_+] \subset A \cup B,$$

so $\sigma_0((v_+, w_+)\triangle(v'_+, w'_+)) \leq \sigma_0(A) + \sigma_0(B) < \frac{\varepsilon}{8C}$ (Figure 5.2).

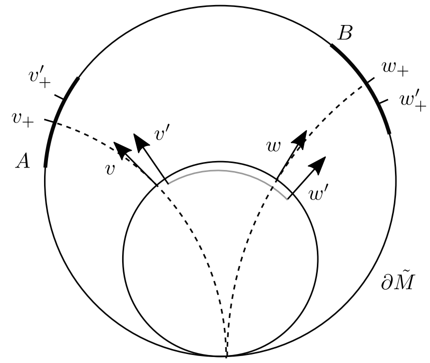

Figure 5.2: Positions of the endpoints of the vectors v, w, v' and w'.

Continuity of the integrated function uniform with respect to η. First, we apply the inner regularity of σ_0: there exists a compact subset $F \subset (v_+, w_+)\setminus\tilde{S}_+$ such that $\sigma_0(((v_+, w_+) \setminus \tilde{S}_+) \setminus F) < \varepsilon/8C$.

If $v' \in T^1\tilde{M}$ and $\eta \in P_{\tilde{H}^u(v')}(\tilde{H}^u(v') \cap \tilde{R}_1)$ then $(v'_-, \eta) \in E_1(\tilde{M})$. Since $F \subset (v_+, w_+) \setminus \tilde{S}_+ \subset P_{\tilde{H}^u(v)}(\tilde{H}^u(v) \cap \tilde{R}_1)$, one has that $\{v_-\} \times F$ is a subset of $E_1(\tilde{M})$. The subset $E^1(\tilde{M})$ is open in $\partial\tilde{M} \times \partial\tilde{M}$ and $\{v_-\} \times F$ is compact, so there exists a neighborhood A of v_- in $\partial\tilde{M}$ such that $A \times F \subset E_1(\tilde{M})$. Let U be a neighborhood of v such that U_+ is contained in A. Now there is a well defined map

$$\begin{array}{ccc} U \times F & \longrightarrow & T^1\tilde{M} \\ (v', \eta) & \longmapsto & P^{-1}_{\tilde{H}^u(v')}(\eta) = P^{-1}(v'_-, \eta, \beta_{v'_-}(0, \pi(v'))), \end{array}$$

which is continuous, because P^{-1} is continuous on $E_1(\tilde{M}) \times \mathbb{R}$ and the Busemann function depends continuously on its three variables. Composing by f we obtain a continuous map from $U \times F$ to $\mathbb{R}$. Since F is compact, this map is continuous at v uniformly with respect the second variable: there exists a neighborhood $U_0 \subset U$ of v such that, for all $v' \in U_0$, for all $\eta \in F$,

$$|f(P^{-1}_{\tilde{H}^u(v')}(\eta)) - f(P^{-1}_{\tilde{H}^u(v)}(\eta))| < \frac{\varepsilon}{2\sigma_0((v_+, w_+))}.$$

Conclusion. We choose a subset F of $(v_+, w_+) \setminus \tilde{S}_+$ as explained above. Thanks to the integral expressions of the measures we can write, for $v' \in T^1\tilde{M}$ and $w' \in \tilde{H}^u(v')$,

$$|\mu_{\tilde{H}^u(v)}((v, w)) - \mu_{\tilde{H}^u(v')}((v', w'))| \leq \int_{F\cap(v'_+, w'_+)} |f(P^{-1}_{\tilde{H}^u(v)}(\eta)) - f(P^{-1}_{\tilde{H}^u(v')}(\eta))|\,\mathrm{d}\sigma_0(\eta) \qquad (*)$$

$$+ \int_{(((v_+,w_+)\setminus F)\cup((v_+,w_+)\triangle(v'_+,w'_+)))\setminus\tilde{S}_+} (|f(P^{-1}_{\tilde{H}^u(v)}(\eta))| + |f(P^{-1}_{\tilde{H}^u(v')}(\eta))|)\,\mathrm{d}\sigma_0(\eta). \qquad (**)$$

We can now see that both terms are small if (v', w') is close to (v, w).

We set $V = U_0 \cap V_1 \cap V_2$ and $W = W_1 \cap W_2$. Let $v' \in V$ and $w' \in W \cap \tilde{H}^u(v')$, so that we can apply all the bounds found above. In the first integral $(*)$, the integrand is bounded by $\frac{\varepsilon}{2\sigma_0((v_+,w_+))}$ and the integrating set has measure at most $\sigma_0((v_+, w_+))$ (we assume $\sigma_0((v_+, w_+)) > 0$ because if $\sigma_0((v_+, w_+)) = 0$ the integral is trivially 0). In $(**)$, the integrand is bounded by $2C$ and the measure of the integrating set is at most $\sigma_0((v_+, w_+) \setminus F) + \sigma_0((v_+, w_+) \triangle (v'_+, w'_+)) < \varepsilon/4C$. So the result is less than ε.

$\square$

5.1.3 Definition of the horocyclic flow on a certain subset of T^1M

Next, we define a subset of the unit tangent bundle $T^1\tilde{M}$ of $\tilde{M}$ and we study the properties of its horocycles and their associated measures. Let $\tilde{\Sigma}_0 \subset T^1\tilde{M}$ denote the set of vectors whose horocycle contains a rank 1 recurrent vector, that is to say,

$$\tilde{\Sigma}_0 = \bigcup_{v \in \tilde{Rec}^1} \tilde{H}^u(v).$$

This set is invariant under Γ, under the geodesic flow and under the horocyclic foliation, in the sense that $\tilde{\Sigma}_0$ contains a horocycle H as soon as it contains one vector of H. Our set $\tilde{\Sigma}_0$ contains a G_δ-dense set, namely the set of rank 1 recurrent vectors Rec^1. The latter is the intersection of the set of rank 1 vectors, which is open and dense [Bal95, Corollary III.3.8], with the set of recurrent vectors, which is G_δ-dense when all the vectors of T^1M are nonwandering. The set $\tilde{\Sigma}_0$ also has full μ_{BM}-measure. By Lemma 4.2.2, all the vectors in $\tilde{\Sigma}_0$ have rank 1 and each horocycle $H \subset \tilde{\Sigma}_0$ coincides with the unstable manifold. This also implies that the horocycles in $\tilde{\Sigma}_0$ do not contain any flat pieces of horocycle.

First, we state and prove a few properties of individual measures on horocycles that later will help us define the parametrization.

Lemma 5.1.2. *Let H be a horocycle of $T^1\tilde{M}$ and $v \in H$:*

(i) The measure μ_H has no point masses.

(ii) If H does not contain any flat piece, then μ_H is of full support in H.

(iii) The measure μ_H is finite on compact sets.

(iv) If v is in $\tilde{\Sigma}_0$, then the half horospheres $\tilde{H}^u_R(v)$ and $\tilde{H}^u_L(v)$ have infinite measure.

Proof. (i) We know that σ_0 has no point masses. If $w \in H$, $\sigma_0(\{v_+\}) = 0$ directly implies that $\mu_H(\{v\}) = 0$.

(ii) If $U \subset H$ is an open nonempty subset, $P_H(U)$ is also open and nonempty. So $\sigma_0(P_H(U)) > 0$ because its support is $\Lambda = \partial\tilde{M}$. Then $\mu_H(U) = \int_{P_H(U)} \phi_v \mathrm{d}\sigma_0 > 0$.

(iii) If $K \subset H$ is compact, $P_H(K)$ is also compact. The function ϕ_v is bounded on $P_H(K)$. The volume part of the integral is bounded by the length of K. Then it is clear that $\mu_H(K)$ is finite.

(iv) By (iii) it is clear that, for every $w \in \tilde{H}^u(v)$, the measure of $\tilde{H}^u_R(v)$ is infinite if and only if the measure of $\tilde{H}^u_R(w)$ is also infinite. So we can assume that v is in $\tilde{Rec}^1$.

Let $B^u(w,r)$ denote the open ball in $\tilde{H}^u(w)$ of center w and radius $r > 0$. The balls $B^u(w,1)$ have two boundary points $a_w, b_w \in \tilde{H}^u(w)$ that depend continuously on w so that $B^u(w,1) = (a_w, b_w)$. In view of Proposition 5.1.1, the function $w \mapsto \mu_{\tilde{H}^u(w)}((a_w,w))$ is continuous. The continuity at v implies that there exists a neighborhood U of v in $\tilde{\Sigma}_0$ such that for all $w \in U$

$$\mu_{\tilde{H}^u(w)}((a_w,w)) \geq \frac{1}{2}\mu_{\tilde{H}^u(v)}((a_v,v)). \tag{5.1}$$

The inequality is in fact valid on $\cup_{\gamma\in\Gamma}\gamma U$ because the family of measures is Γ-invariant.

Since v is recurrent, there is a sequence t_k converging to $-\infty$ and isometries $\gamma_k \in \Gamma$ such that the the distance $d_1(g_{t_k}v, \gamma_k v)$ goes to 0. For k big enough, the vector $g_{t_k}v$ is in $\gamma_k U$, so Equation 5.1 remains true if we replace w by $g_{t_k}v$. Let a_k, b_k be the points in $\tilde{H}^u(g_{t_k}v)$ such that $B^u(g_{t_k}v, 1) = (a_k, b_k)$. Using the fact that the measures on horocycles expand exponentially, we obtain

$$\mu_{\tilde{H}^u(v)}((g_{-t_k}a_k, v)) = e^{-t_k}\mu_{\tilde{H}^u(g_{t_k}v)}((a_k, g_{t_k}v)) \geq \frac{1}{2}e^{-t_k}\mu_{\tilde{H}^u(v)}((a_v,v)).$$

This shows that in one half-horocycle there are subsets of arbitrarily large measure. We proceed analogously for the other half-horocycle, with b_k instead of a_k. $\square$

We can now define a suitable parametrization of the horocyclic flow on the set $\tilde{\Sigma}_0$. Given $v \in \tilde{\Sigma}_0$, we consider the function $m_v : \tilde{H}^u(v) \to \mathbb{R}$ defined by

$$m_v(w) := \begin{cases} \mu_{\tilde{H}^u(v)}((v,w)) & \text{if } w \in \tilde{H}^u_R(v), \\ 0 & \text{if } w = v, \\ -\mu_{\tilde{H}^u(v)}((w,v)) & \text{if } w \in \tilde{H}^u_L(v). \end{cases}$$

The map m_v is well defined by properties (ii) and (iii) of Lemma 5.1.2, is continuous by (i), strictly increasing (with the order given by the orientation) by (ii) and surjective by (iv). Then m_v is in fact a homeomorphism, because the domain and the codomain of the function are topologically the real line.

Definition 5.1.1. We define a horocyclic flow $h_s : \tilde{\Sigma}_0 \to \tilde{\Sigma}_0$ by

$$\forall v \in \Sigma_0, \forall s \in \mathbb{R}, \quad h_s(v) = m_v^{-1}(s).$$

It is clear that h_s satisfies the group law, $h_{s_1} \circ h_{s_2} = h_{s_1+s_2}$, because of the additivity of the measure and property (i). For the same reasons, the measure of every interval $I \subset H^u(v)$ (hence, every measurable set) is preserved, $\mu_{H^u(v)}(h_s(I)) = \mu_{H^u(v)}(I)$. Thanks to the product structure of the measure (Equation 3.3), we deduce that h_s preserves μ_{BM}. The expanding property of the measures on horocycles is transformed into the commutation relation $g_t \circ h_s = h_{se^{\delta t}} \circ g_t$ between the geodesic flow and the horocyclic flow. The Γ-invariance of the measures implies the Γ-invariance of h_s. Only the continuity of h_s remains to be proven.

Lemma 5.1.3. *The flow*

$$\begin{array}{ccc} \mathbb{R} \times \tilde{\Sigma}_0 & \longrightarrow & \tilde{\Sigma}_0 \\ (s, v) & \longmapsto & h_s(v) \end{array}$$

is continuous.

Proof. Let $s \in \mathbb{R}$ and $v \in \tilde{\Sigma}_0$ and consider sequences $s_k \to s$ and $v_k \to v$. We know that the horocycles $\tilde{H}^u(w)$ depend continuously on w, so for each k there exists a vector $w_k \in \tilde{H}^u(v_k)$ such that the sequence $\{w_k\}_k$ converges to $h_s(v)$. By Proposition 5.1.1, we have $\mu_{\tilde{H}^u(v_k)}((v_k, w_k)) \to \mu_{\tilde{H}^u(v)}((v, h_s(v))) = |s|$. We deduce then that the measures of the intervals $(w_k, h_{s_k}(v_k))$ tend to 0. If the distance between w_k and $h_{s_k}(v_k)$ tends to 0 too, then we obtain $h_{s_k}(v_k) \to h_s(v)$ so the flow is continuous at (s, v).

Otherwise, we get a contradiction. To see this, suppose that, for some $\varepsilon > 0$ and subsequence k_i, the Riemannian distance $d_1(w_{k_i}, h_{s_{k_i}}(v_{k_i}))$ is greater than ε. Then, since $w_{k_i} \to h_s(v)$, for i big enough $h_{s_{k_i}}(v_{k_i})$ is at distance greater than $\varepsilon/2$ from $h_s(v)$. But the sequence $h_{s_{k_i}}(v_{k_i})$ must accumulate at some point ζ in $\tilde{H}^u(v) \cup \{v_-\}$, outside of a ball centered at $h_s(v)$. Again by the continuity of the measure, it follows that $\mu_{\tilde{H}^u(v)}((h_s(v), \zeta)) = 0$, which is impossible because the interval is nonempty. $\square$

5.1.4 Unique ergodicity of the horocyclic flow on Σ_0

To study the ergodic properties of the horocyclic flow we introduce the Birkhoff averages. Let $f : T^1M \to \mathbb{R}$ be a Borel function and $\tilde{f} : T^1\tilde{M} \to \mathbb{R}$ its lift. For a number $R > 0$ and v in $\tilde{\Sigma}_0$, we define

$$M_R(f)(v) := \frac{1}{R} \int_0^R \tilde{f}(h_s(v))ds.$$

A simple computation using the commutation relation between the geodesic and the horocyclic flow shows that $M_R(f \circ g_t) = M_{Re^{\delta t}}(f) \circ g_t$.

Moreover, if we assume that f is bounded and uniformly continuous, the equidistribution under the action of the geodesic flow we showed in Theorem A implies that the Birkhoff averages $M_1(f \circ g_t)$ converge pointwise to

$$\frac{1}{\mu_{BM}(T^1M)} \int f \mathrm{d}\mu_{BM}$$

when the time t goes to $+\infty$. However, we need to understand the behavior of $M_R(f)$ when R goes to infinity, that is to say, the equidistribution of horocycles in length. To do this we will use the relation $M_1(f \circ g_t) = M_{e^{\delta t}}(f) \circ g_t$ and some kind of uniform convergence of the averages $M_1(f \circ g_t)$ towards the average of f on the unit tangent bundle of the manifold M, which we are going to prove.

It is clear from the continuity of the measures on horocycles (Proposition 5.1.1) that the function $M_1(f \circ g_t)$ is continuous on $\tilde{\Sigma}_0$. We can prove the following improved result.

Proposition 5.1.4. *Let M be an orientable rank 1 complete connected Riemannian surface with nonpositive curvature satisfying the duality condition. Let f be a bounded and uniformly continuous function on T^1M. Then the family of functions $\{M_1(f \circ g_t)\}_{t>0}$ is equicontinuous at every vector of $\tilde{\Sigma}_0$.*

Proof. Let v be a vector in $\tilde{\Sigma}_0$. The average of the horocyclic flow can be written explicitly as

$$M_1(f \circ g_t)(w) = \int_{(w,h_1(w))} \tilde{f} \circ g_t \,\mathrm{d}\mu_{\tilde{H}^u(w)} =$$

$$= \int_{(w_+,h_1(w)_+)} \tilde{f} \circ g_t(P^{-1}_{\tilde{H}^u(w)}(\eta)) e^{\delta\beta_\eta(0,\,\pi(P^{-1}_{\tilde{H}^u(w)}(\eta)))} \mathrm{d}\sigma_0(\eta). \tag{5.2}$$

Fix $\varepsilon > 0$. We consider a relatively compact neighborhood U of v such that, for all $w \in U$,

$$\sigma_0((v_+, h_1(v)_+) \triangle (w_+, h_1(w)_+)) < \varepsilon.$$

Let C be a uniform bound of $\exp(\delta\beta_\eta(0,\, P^{-1}_{\tilde{H}^u(w)}(\eta)))$ for $w \in U$ and η in a compact neighborhood of $\overline{(v_+, h_1(v)_+)}$. When w approaches v, the set $(w_+, h_1(w)_+)$ will be contained in this compact neighborhood. Then we can change the domain of integration in Equation 5.2 to $(v_+, h_1(v)_+)$ with an error of $\varepsilon \|f\|_\infty C$ at most.

By the uniform continuity of $\tilde{f}$, there is a number $r > 0$ such that $|\tilde{f}(w) - \tilde{f}(w')| < \varepsilon$ if $d_1(w, w') < r$. If w is close enough to v, for all $\eta \in (v_+, h_1(v)_+)$, $P^{-1}_{\tilde{H}^u(w)}(\eta)$ is at distance less than r from $P^{-1}_{\tilde{H}^u(v)}(\eta)$. Applying the geodesic flow g_t, $t \geq 0$ to these two vectors, their distance does not increase. Hence, when w is close to v,

$$\forall t \geq 0,\ \forall \eta \in (v_+, h_1(v)_+), \quad |\tilde{f}(g_t(P^{-1}_{\tilde{H}^u(w)}(\eta))) - \tilde{f}(g_t(P^{-1}_{\tilde{H}^u(v)}(\eta)))| < \varepsilon.$$

This is essentially the same as we did in property (ii) of the proof of Proposition 4.2.1. We can also control the difference between the quantity $\exp(\delta\beta_\eta(0,\, P^{-1}_{\tilde{H}^u(w)}(\eta)))$ and $\exp(\delta\beta_\eta(0,\, P^{-1}_{\tilde{H}^u(v)}(\eta)))$ if we consider a w close enough to v. So the values of the functions at w are close to the values at v uniformly in t when the two vectors are close. This shows that $\{M_1(f \circ g_t)\}_{t>0}$ is equicontinuous at v. □

Both the set $\tilde{\Sigma}_0$ and the functions $M_R(f) : \tilde{\Sigma}_0 \to \mathbb{R}$ are invariant under Γ, so they descend respectively to a set $\Sigma_0 \subset T^1M$ and some functions $\bar{M}_R(f) : \Sigma_0 \to \mathbb{R}$. The family of functions is still $\bar{M}_1(f \circ g_t)$ equicontinuous. In the next proposition, we combine this with the equidistribution of horocycles to show that $\bar{M}_1(f \circ g_t)$ converges uniformly on compact subsets of Σ_0.

Proposition 5.1.5. *Let M be an orientable rank 1 complete connected Riemannian surface with nonpositive curvature satisfying the duality condition. Assume that the Bowen-Margulis measure μ_{BM} is finite. Let f be a bounded and uniformly continuous function on T^1M. Then the functions $\bar{M}_1(f \circ g_t) : \Sigma_0 \to \mathbb{R}$ converge uniformly on compact sets to the constant $\int f \mathrm{d}\mu_{BM}/\mu_{BM}(T^1M)$ when the time t tends to $+\infty$.*

Proof. The functions $M_1(f \circ g_t)$ are bounded by the uniform norm $\|f\|_\infty$. Let K be a subset of Σ_0. We apply the Arzelà-Ascoli theorem on the space of continuous functions $C(K)$. For every uniformly continuous and bounded function $f : T^1M \to \mathbb{R}$, the family $\{\bar{M}_1(f \circ g_t)|_K\}_{t>0} \subset C(K)$ is equicontinuous and uniformly bounded, so it is a relatively compact subset of $C(K)$ in the uniform convergence topology. On the other hand, $\bar{M}_1(f \circ g_t)$ converges pointwise on K to $\int f \mathrm{d}\mu_{BM}/\mu_{BM}(T^1M)$ when $t \to +\infty$. This is enough to conclude that the convergence is uniform, as we recall below.

Lemma 5.1.6. *Let $\{f_n\}$ be a family of functions relatively compact in the uniform topology. Assume that f_n converges pointwise to a function $\bar{f}$. Then, f_n converges uniformly to $\bar{f}$.*

Proof. Assume that f_n does not converge uniformly to $\bar{f}$. Then $\{f_n\}$ has an accumulation point g different from $\bar{f}$, so we can write $g = \lim f_{n_k}$ for a subsequence n_k. In particular, f_{n_k} converges pointwise to g. By uniqueness of the limit, g and $\bar{f}$ are equal, which contradicts the assumption. $\square$

$\square$

Finally, we state Theorem B again and prove it with the help of the Birkhoff averages. Recall that Σ_0 is the set of vectors whose horosphere contains a rank 1 recurrent vector, and Σ_0 has full μ_{BM}-measure in T^1M.

Theorem 5.1.7. *Let M be an orientable rank 1 complete connected Riemannian surface with nonpositive curvature satisfying the duality condition. Assume that the Bowen-Margulis measure μ_{BM} is finite. Then every finite Borel measure on Σ_0 invariant under the horocyclic flow h_s is a constant multiple of the Bowen-Margulis measure $\mu_{BM}|_{\Sigma_0}$ restricted to Σ_0.*

Proof. Firstly, let us prove that, for every bounded and uniformly continuous function $f : T^1M \to \mathbb{R}$ and for every vector v in Σ_0, there exists a sequence $t_n \to +\infty$ such that the Birkhoff integral $\bar{M}_{e^{t_n}}(f)(v)$ tends to $\lambda := \int f \mathrm{d}\mu_{BM}/\mu_{BM}(T^1M)$.

There is a recurrent vector w in the unstable horocycle of v, since v is in Σ_0. Let t_n be a sequence tending to $+\infty$ such that $g_{-t_n}(w) \to w$. Then obviously $g_{-t_n}(v)$ also tends to w. We consider the compact set $K := \{g_{-t_n}(v)\}_{n\geq 0} \cup \{w\} \subset \Sigma_0$. By Proposition 5.1.5, the functions $\bar{M}_1(f \circ g_t)$ converge uniformly on K to the global average λ of f. Therefore, using the time-scale relation, we have

$$|\bar{M}_{e^{t_n}}(f)(v) - \lambda| = |\bar{M}_1(f \circ g_{t_n})(g_{-t_n}(v)) - \lambda| \leq \sup_{u\in K} |\bar{M}_1(f \circ g_{t_n})(u) - \lambda| \xrightarrow{n\to+\infty} 0.$$

We can now prove that the restriction $\mu_{BM}|_{\Sigma_0}$ of μ_{BM} to Σ_0 is the unique measure on Σ_0 invariant under h_s, up to a multiplicative constant. Suppose that ν is an ergodic h_s-invariant probability measure on Σ_0. By the Birkhoff ergodic theorem, for every bounded and uniformly continuous function $f : \Sigma_0 \to \mathbb{R}$, for ν-a.e. v in Σ_0, we have

$$\bar{M}_R(f)(v) = \frac{1}{R}\int_0^R f(h_s(v))\mathrm{d}s \xrightarrow{R\to+\infty} \int_{\Sigma_0} f\mathrm{d}\nu.$$

We take v one of the points of Σ_0 where $\bar{M}_R(f)$ converges to $\int f\mathrm{d}\nu$. We can extend f to a bounded and uniformly continuous function $\hat{f}$ on T^1M, because Σ_0 is dense in T^1M. As we have seen, there is a sequence $R_n = e^{t_n}$ where $\bar{M}_R(\hat{f})(v) = \bar{M}_R(f)(v)$ tends to λ as well. So we obtain

$$\int_{\Sigma_0} f\mathrm{d}\nu = \lambda = \frac{\int_{T^1M} f\mathrm{d}\mu_{BM}}{\mu_{BM}(T^1M)} = \frac{\int_{\Sigma_0} f\mathrm{d}\mu_{BM}}{\mu_{BM}(\Sigma_0)},$$

because Σ_0 has full μ_{BM}-measure. We have concluded that ν is equal to the normalization of $\mu_{BM}|_{\Sigma_0}$. $\square$

5.1.5 Alternative proof of the unique ergodicity

We would like to point out another way to prove Proposition 5.1.5, which does not require the equidistribution of horocycles (Theorem 4.2.4). Instead, we use a version of the Arzelà-Ascoli theorem for the compact-open topology, the ergodic theorem and the fact that there exists a dense horocycle in Σ_0. Actually, we can prove that all the horocycles of Σ_0 are dense.

Lemma 5.1.8. *Let M be a rank 1 nonpositively curved complete connected Riemannian surface with the duality condition. Then every horocycle H contained in Σ_0 is dense in T^1M.*

Proof. This follows directly from two results of Eberlein. A nonpositively curved complete connected manifold that satisfies the visibility axiom and the duality condition, like M, has a dense horocycle in T^1M [Ebe73a, Theorem 5.2]. Next we apply [Ebe73a, Theorem 5.5] to M, which says that a horocycle $H^u(v)$ is dense in T^1M if and only if v is not almost minimizing. We say that v is *almost minimizing* if there exists a constant $C > 0$ such that, for all $t \geq 0$, we have $d(\pi(v), \pi(g_t(v))) \geq t - C$. If a horocycle H is contained in Σ_0, then there is a recurrent vector in H and, in particular, this vector is not almost minimizing. Thus, the horocycle H is dense in T^1M □

We consider the space of continuous functions $C(\Sigma_0)$ on the set Σ_0 equipped, this time, with the compact-open topology. Recall that, for functions on a metric space, the convergence in the compact-open topology is equivalent to the uniform convergence on compact subsets.

Proof of Proposition 5.1.5. Let f be a bounded and uniformly continuous function on T^1M. Applying the Arzelà-Ascoli theorem for the compact-open topology [Dug66, Theorem XII.6.4], since the family of functions $\{\bar{M}_1(f \circ g_t)\}_{t>0}$ is equicontinuous and uniformly bounded, we obtain that it has a compact closure in $C(\Sigma_0)$ endowed with the compact-open topology. To complete the proof, we show that the only accumulation point of $\{\bar{M}_1(f \circ g_t)\}_{t>0}$ in the compact-open topology is the constant function $\int f \mathrm{d}\mu_{BM}/\mu_{BM}(T^1M)$.

Let φ in $C(\Sigma_0)$ be the limit of a sequence $\bar{M}_1(f \circ g_{t_k})$ in the compact-open topology, where $t_k \to +\infty$. By the dominated convergence theorem, φ is the limit in $L^2(\Sigma_0, \mu_{BM}|_{\Sigma_0})$ of the same sequence. On the other hand, we apply the L^2 ergodic theorem for the system $(\Sigma_0, h_s, \mu_{BM}|_{\Sigma_0})$ to the function $f \in L^2(\Sigma_0, \mu_{BM}|_{\Sigma_0})$. We conclude that $\bar{M}_t(f)$ converges to an h_s-invariant function $\bar{f}$ in the L^2 norm, with the equality $\int \bar{f} \mathrm{d}\mu_{BM} = \int f \mathrm{d}\mu_{BM}$. Thanks to the g_t invariance of μ_{BM}, we have the inequality

$$\left\|\varphi - \bar{f} \circ g_{t_k}\right\|_2 \leq \left\|\varphi - \bar{M}_1(f \circ g_{t_k})\right\|_2 + \left\|\bar{f} - \bar{M}_{e^{t_k}}(f)\right\|_2,$$

which implies the L^2-convergence of $\bar{f} \circ g_{t_k}$ to φ, because both terms on the right side tend to zero. The function $\bar{f}$ is a.e. h_s-invariant, so $\bar{f} \circ g_{t_k}$ are also a.e. invariant because the geodesic and the horocyclic flow commute. Then their limit, the continuous function φ defined on Σ_0, is invariant under h_s.

In brief, the function φ is constant on the orbits of h_s, and these orbits are dense by Lemma 5.1.8. Since φ is continuous, we conclude that it is constant on Σ_0. In fact, the value of the constant is $\int f \mathrm{d}\mu_{BM}/\mu_{BM}(T^1M)$, because we have

$$\int \varphi \, \mathrm{d}\mu_{BM} = \int \bar{f} \circ g_{t_k} \, \mathrm{d}\mu_{BM} = \int \bar{f} \, \mathrm{d}\mu_{BM} = \int f \, \mathrm{d}\mu_{BM}.$$

□

5.1.6 Final remark

Theorem 5.1.7 does not solve completely the problem of the horocyclic flow in nonpositive curvature, since it does not say what happens to the flow outside the set Σ_0. For instance, we wonder if a horocyclic flow defined everywhere on a compact nonpositively curved surface is uniquely ergodic. We will study this question for the class of compact manifolds without flat strips in Section 5.3.

5.2 Reparametrization of the horocyclic flow

In this section we will study how invariant measures are transformed by a change of parametrization. This will eventually allow us to conclude that the unique ergodicity also holds for the arc-length parametrization. We start by a general result on reparametrization of measures.

5.2.1 A general reparametrization result

Let X be a topological space. A flow on X is a map

$$\begin{array}{ccc} X \times \mathbb{R} & \longrightarrow & X \\ (x,t) & \longmapsto & f_t(x). \end{array}$$

satisfying for all $x \in X$ and $t_1, t_2 \in \mathbb{R}$, $f_0(x) = x$ and $f_{t_1+t_2}(x) = f_{t_1}(f_{t_2}(x))$. By abuse of notation, we denote the flow by f_t. We say that the flow f_t is continuous if the map $(x,t) \mapsto f_t(x)$ is continuous. The orbit of $x \in X$ is the set $\{f_t(x)\}_{t\in\mathbb{R}}$. The point x is fixed by the flow if its orbit consists of a single point.

Let f_t be a continuous flow on X. A Borel measure μ on X is said to be invariant by f_t if for every Borel subset A of X we have, for all $t \in \mathbb{R}$, $\mu(f_t(A)) = \mu(A)$. Let $\mathrm{Mes}(f_t)$ denote the set of locally finite Borel measures on X invariant by f_t. Let $\mathrm{Prob}(f_t)$ be the subset of $\mathrm{Mes}(f_t)$ consisting of the probability measures. The flow f_t is *uniquely ergodic* if the set $\mathrm{Prob}(f_t)$ consists of a single measure. The next result is due to M. Beboutoff and W. Stepanoff in 1940.

Theorem 5.2.1. *[BS40] Let X be a separable metric space. Let f_t be a continuous flow on X without fixed points. Let g_t be another continuous flow on X with the same orbits than f_t, i.e. for all $x \in X$, we have $\{g_t(x)\}_{t\in\mathbb{R}} = \{f_t(x)\}_{t\in\mathbb{R}}$. Then there is a bijective correspondence $\Phi : \mathrm{Mes}(f_t) \to \mathrm{Mes}(g_t)$.*

The correspondence Φ will be described in the next section in the setting of horocyclic flows. If X is a compact space, since every locally finite measure is in fact finite, the map Φ given in the theorem is an isomorphism between the spaces of finite Borel invariant measures of f_t and g_t. This induces a bijection between $\mathrm{Prob}(f_t)$ and $\mathrm{Prob}(g_t)$. In particular, f_t is uniquely ergodic if and only if g_t is uniquely ergodic.

However, in a general separable metric space, we think that there is no reason why the map of the theorem would send finite measures to finite measures, and infinite measures to infinite measures.

5.2.2 Reparametrization of the horocyclic flow on a compact surface

In this section, we apply the ideas of Beboutoff and Stepanoff [BS40] to establish what happens to a measure invariant by a horocyclic flow under a change of parametrization. We specify the bijection Φ for a horocyclic flow on a compact surface.

In what follows, let M denote an oriented compact connected rank 1 surface with nonpositive curvature. Such a manifold M is complete, has a rank 1 closed geodesic, and the covering transformations group Γ is non-elementary and divergent [Kni98, Theorem 4.3]. Moreover, the Bowen-Margulis measure we defined in Section 3.3 is finite.

By *horocyclic flow* we mean a continuous flow h_s on T^1M whose orbits are the unstable horocycles, i.e

$$\forall v \in T^1M,\ \{h_s(v)\}_{s\in\mathbb{R}} = H^u(v).$$

Definition 5.2.1. The Lebesgue horocyclic flow h_s^L is given by the arc length of the horocycles: for any $v \in T^1M$ and $s \in \mathbb{R}$, the vector $h_s^L(v)$ is the vector of $H^u(v)$ that we get by traveling a distance $|s|$ on $H^u(v)$ from v in the positive or negative direction accordingly to the orientation depending on the sign of s. We will say that the horocyclic flow h_s^L has the *Lebesgue parametrization.*

A horocyclic flow h_s on T^1M is lifted to a horocyclic flow $\tilde{h}_s$ on the unit tangent bundle $T^1\tilde{M}$ of the universal cover satisfying $\tilde{h}_s(\gamma v) = \gamma\tilde{h}_s(v)$ for every $s \in \mathbb{R}$, $v \in T^1\tilde{M}$, $\gamma \in \Gamma$. Conversely, a horocyclic flow $\tilde{h}_s$ on $T^1\tilde{M}$ with the property $\tilde{h}_s(\gamma v) = \gamma\tilde{h}_s(v)$ passes to the quotient T^1M giving a horocyclic flow h_s. Moreover, h_s-invariant measures on T^1M are in correspondence with $\tilde{h}_s$-invariant measures on $T^1\tilde{M}$ which in addition are Γ-invariant.

The *weak unstable manifold* $\tilde{W}^{wu}(v)$ of a vector $v \in T^1\tilde{M}$ is the union of all horocycles along the geodesic generated by v,

$$\tilde{W}^{wu}(v) = \cup_{t\in\mathbb{R}} H(g_t v) = \{w \in T^1\tilde{M} \mid w_- = v_-\},$$

and has dimension 2. In the universal cover, the *weak stable manifold* $\tilde{W}^{ws}(v) = -\tilde{W}^{wu}(-v)$ of a vector $v \in T^1\tilde{M}$ is a section of the flow in the sense of [BS40] in many cases, as explained in the next lemma.

Lemma 5.2.2. *Let $v \in T^1\tilde{M}$. Assume that the weak stable manifold $\tilde{W}^{ws}(v)$ of v has no rank 2 vectors. Then for every $w \in T^1\tilde{M} \setminus \tilde{W}^{wu}(-v) = \{u \in T^1\tilde{M} \mid u_- \neq v_+\}$, there exists a unique time $s \in \mathbb{R}$ such that $\tilde{h}_s(w) \in \tilde{W}^{ws}(v)$.*

Proof. It is known that a nonflat compact surface M with nonpositive curvature satisfies the visibility axiom, which means that any two distinct points of the boundary $\partial\tilde{M}$ can be joined by a geodesic on $\tilde{M}$ [Ebe79, Proposition 2.5]. If v and w are as in the statement, the points v_+ and $w_- \in \partial\tilde{M}$ are distinct, so there is at least a geodesic between them. Hence, there exists a vector u in the unstable horocycle $\tilde{H}^u(w)$ of w pointing to v_+. This vector can be written as $u = \tilde{h}_s(w)$ for some $s \in \mathbb{R}$ and is in $\tilde{W}^{ws}(v)$.

To prove the uniqueness, let us suppose that for some different reals s and s', $\tilde{h}_s(w)$ and $\tilde{h}_{s'}(w)$ are in the weak stable manifold $\tilde{W}^{ws}(v)$. Then the vectors $\tilde{h}_s(w)$ and $\tilde{h}_{s'}(w)$ are asymptotic both for positive and negative time, so the corresponding geodesics bound a flat strip. This would imply that these vectors have rank 2, which contradicts the hypothesis that $\tilde{W}^{ws}(v)$ has no such vectors. $\square$

The condition that $\tilde{W}^{ws}(v)$ has no rank 2 vectors is equivalent to the fact that v_+ is not in the set $\tilde{S}_+$ of endpoints of vectors of rank 2. We know that $\tilde{S}_+$ has zero σ_0-measure, so its complement must be dense in $\partial\tilde{M}$. This ensures that there are enough sections for the horocyclic flow.

We recall how an invariant measure is locally disintegrated. Let $\mu \in \mathrm{Mes}(\tilde{h}_s)$ be an invariant measure. Given a Borel subset A of a section $\tilde{W}^{ws}(v)$, we consider the function $\phi_A : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ defined by

$$\phi_A(s) = \mu(\tilde{h}_{[0,s]}(A)).$$

Let us assume that $\phi_A(1)$ is finite. Then we have, for any integer number $n \geq 1$,

$$\begin{aligned}\phi_A(0) = \mu(A) \leq \mu(\tilde{h}_{[0,1/n)}(A)) &= \frac{1}{n}\sum_{k=0}^{n-1} \mu(\tilde{h}_{k/n}(\tilde{h}_{[0,1/n)}(A))) \\ &= \frac{1}{n}\mu(\tilde{h}_{[0,1)}(A)) \leq \frac{\phi_A(1)}{n},\end{aligned}$$

hence $\phi_A(0) = 0$. Moreover, for every two nonnegative numbers s, t such that $s + t \leq 1$, we have

$$\phi_A(s+t) = \phi_A(s) + \mu(h_s(h_{(0,t]}(A))) = \phi_A(s) + \phi_A(t) - \phi_A(0) = \phi_A(s) + \phi_A(t)$$

thanks to the invariance of μ. Since ϕ_A is monotonic, we deduce that it is linear, so there is a constant $l_A \geq 0$ such that $\phi_A(t) = l_A t$ for all $t \in [0,1]$.

We now define a measure $\mu_{\tilde{W}^{ws}(v)}$ on $\tilde{W}^{ws}(v)$ which associates the value $l_A = \phi_A(1)$ to the set A if $\phi_A(1)$ is finite, and the value ∞ otherwise. It is not difficult to check that $\mu_{\tilde{W}^{ws}(v)}$ is a Borel locally finite measure and it is the same for two vectors on the same weak stable leaf. Furthermore, the measure μ is the product of the Lebesgue measure on each horocycle by the measure $\mu_{\tilde{W}^{ws}(v)}$ (Figure 5.3): for every Borel subset $E \subseteq T^1\tilde{M} \setminus \tilde{W}^{wu}(-v)$ we have

$$\mu(E) = \int_{\tilde{W}^{ws}(v)} \int_{\mathbb{R}} \mathbf{1}_E(\tilde{h}_s(u))\,\mathrm{d}s\,\mathrm{d}\mu_{\tilde{W}^{ws}(v)}(u).$$

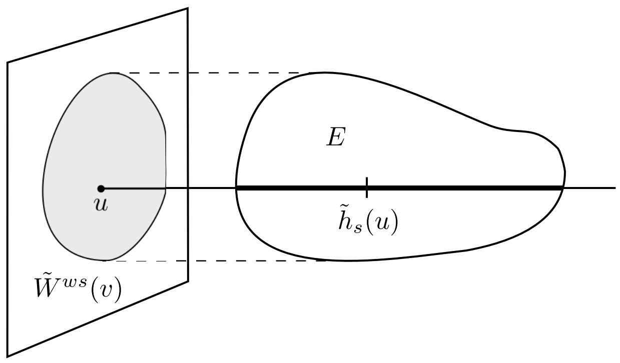

Figure 5.3: Decomposition of μ.

Let h'_s be another horocyclic flow of T^1M. The two horocyclic flows $\tilde{h}_s$ and $\tilde{h}'_s$ on $T^1\tilde{M}$ are related by a change of time, given $v \in T^1\tilde{M}$ and $s' \in \mathbb{R}$, there exists a unique $s = s(s', v) \in \mathbb{R}$ such that $\tilde{h}_s(v) = \tilde{h}'_{s'}(v)$. In fact, s as a function $\mathbb{R} \times T^1\tilde{M} \to \mathbb{R}$ satisfies:

(i) For all $v \in T^1\tilde{M}$ and $s' \in \mathbb{R}$, we have $\tilde{h}'_{s'}(v) = \tilde{h}_{s(s',v)}(v)$.

(ii) s is continuous.

(iii) For all $v \in T^1\tilde{M}$, $s(\,\cdot\,, v) : \mathbb{R} \to \mathbb{R}$ is strictly monotonic.

(iv) For all $v \in T^1\tilde{M}$, $s(0, v) = 0$.

(v) For all $v \in T^1\tilde{M}$ and $s'_1, s'_2 \in \mathbb{R}$, $s(s'_1 + s'_2, v) = s(s'_1, v) + s(s'_2, \tilde{h}_{s(s'_1,v)}(v))$.

(vi) For all $v \in T^1\tilde{M}$, $s' \in \mathbb{R}$ and $\gamma \in \Gamma$, $s(s', \gamma v) = s(s', v)$

The converse is also true: given a function s with the properties (ii)-(vi) we can define a new horocyclic flow by $\tilde{h}'_{s'}(v) := \tilde{h}_{s(s',v)}(v)$ which passes to the quotient T^1M.

As stated in Theorem 5.2.1, there is a correspondence between the measures invariant by the two flows.

Definition 5.2.2. Given two horocyclic flows $\tilde{h}_s$ and $\tilde{h}'_s$ of $T^1\tilde{M}$, we define a map $\Phi : \mathrm{Mes}(\tilde{h}_s) \to \mathrm{Mes}(\tilde{h}'_s)$ putting, for every weak stable manifold $\tilde{W}^{ws}(v)$ containing no rank 2 vectors and any Borel subset E of $T^1\tilde{M} \setminus \tilde{W}^{wu}(-v)$,

$$(\Phi(\mu))(E) = \int_{\tilde{W}^{ws}(v)} \int_{\mathbb{R}} \mathbf{1}_E(\tilde{h}'_s(u))\, \mathrm{d}s\, \mathrm{d}\mu_{\tilde{W}^{ws}(v)}(u). \tag{5.3}$$

By [BS40], the measure $\mu' = \Phi(\mu)$ is well defined and $\tilde{h}'_s$-invariant. If the measure μ is in addition Γ-invariant, then so is the measure μ'. We obtain, after normalization of measures, a correspondence between the sets $\mathrm{Prob}(h_s)$ and $\mathrm{Prob}(h'_s)$, because T^1M is compact. We also remark that the measures $\mu_{\tilde{W}^{ws}(v)}$ are independent of the parametrization of the horocyclic flow.

5.2.3 Horocyclic flows on a subset of T^1M

Let M be a compact oriented nonpositively curved rank 1 surface. We consider a Borel subset Σ of T^1M which is a union of horocycles and let $\tilde{\Sigma}$ be its lift to $T^1\tilde{M}$. We want to study the continuous flows defined on Σ whose orbits are horocycles. It is clear that a horocyclic flow h_s on T^1M is restricted to a horocyclic flow $h_s|_\Sigma$ on Σ, and that a measure $\mu \in \mathrm{Mes}(h_s)$ can be restricted to a measure $\mu|_\Sigma \in \mathrm{Mes}(h_s|_\Sigma)$. In many situations, a certain parametrization of the horocyclic flow is just defined on a subset Σ and we would like to deduce ergodic properties for the flow on the whole space from this specific parametrization.

The set Σ does not have to be compact, or even locally compact. Then, if we have two horocyclic flows h_s and h'_s on Σ, we still have a bijection between $\mathrm{Mes}(h_s)$ and $\mathrm{Mes}(h'_s)$ by Theorem 5.2.1, but we have no information about the subsets $\mathrm{Prob}(h_s) \subset \mathrm{Mes}(h_s)$ and $\mathrm{Prob}(h'_s) \subset \mathrm{Mes}(h'_s)$ or the relation between them.

We can restrict the sections of the horocyclic foliation that we found in the previous section to $\tilde{\Sigma}$ and disintegrate invariant measures with respect to them. More precisely, if the set $\tilde{W}^{ws}(v) \cap \tilde{\Sigma}$ is nonempty and $\tilde{W}^{ws}(v)$ has no rank 2 vectors, then $\tilde{W}^{ws}(v) \cap \tilde{\Sigma}$ is a section for the flow $\tilde{h}_s$ and we can define a measure $\mu_{\tilde{W}^{ws}(v) \cap \tilde{\Sigma}}$ on this set from a $\tilde{h}_s$-invariant measure μ on $\tilde{\Sigma}$.

Another horocyclic flow $\tilde{h}'_{s'}$ on $\tilde{\Sigma}$ is related to $\tilde{h}_s$ by a change of time $s = s(s', v)$ as before. The $\tilde{h}'_s$-invariant measure $\mu' = \Phi(\mu)$ associated to $\mu \in \mathrm{Mes}(\tilde{h}_s)$ has a local expression given by Equation (5.3) as a product of the Lebesgue measures on horocycles by $\mu_{\tilde{W}^{ws}(v)\cap\tilde{\Sigma}}$.

5.2.4 The Margulis parametrization

Some of the most relevant properties of the horocyclic flow are deduced thanks to the Margulis parametrization, which allows to apply the usual techniques of ergodic theory. In Section 5.1, we established the unique ergodicity of the Margulis horocyclic flow for a certain class of nonpositively curved surfaces. The situation is different from negatively curved manifolds, because the Margulis parametrization can only be defined on a certain subset Σ_0 of T^1M. Here we will study the relation between the Margulis parametrization h_s^M on Σ_0 and the Lebesgue parametrization h_s^L on T^1M. We will see that the map Φ induces a bijection between $\mathrm{Prob}(h_s^L|_{\Sigma_0})$ and $\mathrm{Prob}(h_s^M)$, which is not trivial at all because no assumptions of compactness are made on Σ_0.

As we have seen in Section 3.4, the set of horocycles of $T^1\tilde{M}$ admits a family of measures $\{\mu_H\}_H$, which is exponentially expanded by the geodesic flow,

$$\mu_{g_t H} = e^{\delta t} g_{t*}\mu_H.$$

To define the Margulis parametrization, we parametrize each horocycle H by the measure μ_H. Let us explore a few further properties of these measures.

Lemma 5.2.3. *The measures μ_H are locally finite and have no point masses.*

Proof. The measure μ_H is obtained from the Patterson-Sullivan measure σ_0 on $\partial\tilde{M}$ as

$$\mathrm{d}\mu_H(v) = e^{\delta\beta_{v_+}(0,\pi(v))}\mathrm{d}\sigma_0(v_+).$$

Since σ_0 is finite and the factor is bounded on bounded sets, μ_H is locally finite. We also know that σ_0 has no point masses, so neither does μ_H. □

The orientation of M induces an orientation on each horocycle H, so there are well defined positive and negative directions. A vector $v \in H$ divides the horocycle into two infinite intervals, we write $\tilde{H}^u_R(v)$ for the one in the positive direction and $\tilde{H}^u_L(v)$ for the other. Next, we give some conditions on a subset of Σ that allow us to define the Margulis parametrization.

Proposition 5.2.4. *Let M be a compact oriented nonpositively curved rank 1 surface. Let $\tilde{\Sigma}$ be a Borel subset of $T^1\tilde{M}$ which is a union of horocycles. We assume that for every horocycle $H \subset \tilde{\Sigma}$,*

- *(i) the measure μ_H is of full support in H,*
- *(ii) for one (hence for all) vector $v \in H$, the half horocycles $\tilde{H}^u_R(v)$ and $\tilde{H}^u_L(v)$ have infinite measure.*

Then there exists a horocyclic flow $\tilde{h}_s^M$ on $\tilde{\Sigma}$ such that for all $v \in \tilde{\Sigma}$ and $s \in \mathbb{R}$, we have

$$\mu_{\tilde{H}^u(v)}((v, \tilde{h}_s^M(v))) = |s|.$$

Moreover, the flow $\tilde{h}_s^M$ satisfies, for every Borel subset A of the horocycle $\tilde{H}^u(v)$,

$$\mu_{\tilde{H}^u(v)}(A) = \mathrm{Leb}(\{s \in \mathbb{R} \mid \tilde{h}_s^M(v) \in A\}).$$

Proof. Let H be a horocycle in $\tilde{\Sigma}$ and $v \in H$. We consider the function $m_v : H \to \mathbb{R}$ defined by $m_v(w) = \pm\mu_{\tilde{H}^u(v)}((v,w))$ if $w \in \tilde{H}^u_{R/L}(v)$ and $m_v(v) = 0$. It is continuous because of Proposition 5.1.1, strictly increasing because μ_H has full support in H and is surjective because both half-horocycles have infinite measure. Since H is homeomorphic to $\mathbb{R}$, the map m_v is a homeomorphism. We then define $\tilde{h}^M_s(v) = m_v^{-1}(s)$ for all $s \in \mathbb{R}$.

It is clear that the orbit of v is the horocycle H. Also, μ_H has no point masses by Lemma 5.2.3, so we have $m_{\tilde{h}^M_s(v)}(w) = m_v(w) - s$. This leads to the additive property

$$\tilde{h}^M_s \circ \tilde{h}^M_{s'} = \tilde{h}^M_{s+s'}.$$

To see the equality of measures, we observe that

$$\mu_H((v, \tilde{h}^M_s(v))) = |m_v(\tilde{h}^M_s(v))| = |m_v(m_v^{-1}(s))| = |s|, \tag{5.4}$$

from the definition of the flow. If we take the pullback $m_v^*\,\mathrm{Leb}$ of the Lebesgue measure by m_v, we can see that $m_v^*\,\mathrm{Leb}$ coincides with μ_H on the intervals (w_1, w_2), so they are equal. Actually, if we set $w_1 = \tilde{h}^M_{s_1}(v)$ and $w_2 = \tilde{h}^M_{s_2}(v)$,

$$\begin{aligned}\mathrm{Leb}(m_v((w_1, w_2))) &= \mathrm{Leb}((s_1, s_2)) = |s_2 - s_1| \\ &= \mu_H((w_1, \tilde{h}^M_{s_2 - s_1}(w_1))) = \mu_H((w_1, w_2)),\end{aligned}$$

thanks to (5.4). Then for every Borel subset A of H, $\mu_H(A) = \mathrm{Leb}(m_v(A))$. But $s \in m_v(A)$ if and only if $\tilde{h}^M_s(v) = m_v^{-1}(s) \in A$. This proves the second property.

It remains to prove the continuity of the flow as a map from $\mathbb{R} \times \tilde{\Sigma} \to \tilde{\Sigma}$. Fix a couple $(s, v) \in \mathbb{R} \times \tilde{\Sigma}$ and consider a sequence $((s_k, v_k))_k$ of elements of $\mathbb{R} \times \tilde{\Sigma}$ converging to (s, v). We need to show that $\tilde{h}^M_{s_k}(v_k)$ converges to $\tilde{h}^M_s(v)$. We assume $s \geq 0$, the other case being done analogously. We know that the horocycles $\tilde{H}^u(w)$ depend continuously on w, so for each k there exists a vector $w_k \in \tilde{H}^u(v_k)$ such that the sequence $\{w_k\}_k$ converges to $\tilde{h}^M_s(v)$. By Proposition 5.1.1, we know that $\mu_{\tilde{H}^u(v_k)}((v_k, w_k))$ converges to $\mu_{\tilde{H}^u(v)}((v, \tilde{h}^M_s(v))) = s$ when k tends to infinity. We deduce then that the measures of the intervals $(w_k, \tilde{h}^M_{s_k}(v_k))$ go to 0.

We claim that the distance between w_k and $\tilde{h}^M_{s_k}(v_k)$ tends to 0. Assume, contrary to our claim, that, for some $\varepsilon > 0$, there is a subsequence k_i such that the Riemannian distance $d_1(w_{k_i}, \tilde{h}^M_{s_{k_i}}(v_{k_i}))$ is greater than ε. Let us consider the points $\tilde{h}^L_\varepsilon(w_{k_i})$, which are in the interval $(w_{k_i}, \tilde{h}^M_{s_{k_i}}(v_{k_i}))$. So the $\mu_{\tilde{H}^u(v_{k_i})}$-measure of $(w_{k_i}, \tilde{h}^L_\varepsilon(w_{k_i}))$ also tends to 0. In the limit, we have

$$\mu_{\tilde{H}^u(v)}((\tilde{h}^M_s(v), \tilde{h}^L_\varepsilon(\tilde{h}^M_s(v)))) = 0$$

thanks again to the continuity of the measures. This is a contradiction because $\mu_{\tilde{H}^u(v)}$ has full support in $\tilde{H}^u(v)$ by hypothesis. Finally, since w_k converges to $\tilde{h}^M_s(v)$, the sequence $\tilde{h}^M_{s_k}(v_k)$ also converges to this point.

□

The point of this discussion is the following result that establishes a bijection between the finite invariant measures of the flows $h^L_s|_\Sigma$ and h^M_s, both defined on Σ.

Proposition 5.2.5. *Let $\tilde{\Sigma}$ be a subset of $T^1\tilde{M}$ satisfying the hypothesis of Proposition 5.2.4. Assume that there are at least two distinct stable manifolds of vectors in $\tilde{\Sigma}$ that do not have rank 2 vectors. Consider the horocyclic flow h_s^M on the set $\Sigma \subset T^1M$ there defined. The map $\Phi : \mathrm{Mes}(h_s^L|_\Sigma) \to \mathrm{Mes}(h_s^M)$ sends finite measures to finite measures, and infinite measures to infinite measures.*

Proof. We first show that the image of a finite measure is finite. Let $\mu \in \mathrm{Mes}(\tilde{h}_s^L|_{\tilde{\Sigma}})$ be a Γ-invariant measure whose projection to $\Sigma \subset T^1M$ is finite and let $\mu' = \Phi(\mu)$ be its image in $\mathrm{Mes}(\tilde{h}_s^M)$.

Let $D \subset T^1\tilde{M}$ be a compact fundamental domain for the action of Γ. We want to show that $\mu'(D \cap \tilde{\Sigma})$ is finite. Let v_1, $v_2 \in \tilde{\Sigma}$ two vectors such that $\tilde{W}^{ws}(v_1)$ and $\tilde{W}^{ws}(v_2)$ are distinct sections of the horocyclic flow (i.e. they do not contain any rank 2 vector). Take disjoint open neighborhoods A, B in $\partial\tilde{M}$ of the points v_{1+}, v_{2+}. By the continuity of the projection to the boundary we know that the sets

$$\{w \in D \,|\, w_- \notin A\}, \quad \{w \in D \,|\, w_- \notin B\}$$

are closed in D, therefore compact. They form a cover of D. This reduces the proof to showing that $\mu'(K \cap \tilde{\Sigma})$ is finite where K is a compact set such that $v_{0+} \notin K_-$, where v_0 is any vector of $\tilde{\Sigma}$ whose stable manifold has no rank 2 vectors. Let us fix such a vector v_0 until the end of the proof.

By Lemma 5.2.2, for every $w \in K$ there exists a number $s(w)$ such that $\tilde{h}^L_{s(w)}(w) \in \tilde{W}^{ws}(v_0)$ (Figure 5.4). We see $s(w)$ as a continuous function from K to $\mathbb{R}$. It is then bounded by some constant $S > 0$. The function $w \mapsto \tilde{h}^L_{s(w)}(w)$ is also continuous, so the projection of K to $\tilde{W}^{ws}(v_0)$ is a compact set, denoted by L. The following inclusion holds,

$$K \subset \tilde{h}^L_{[-S,S]}(L).$$

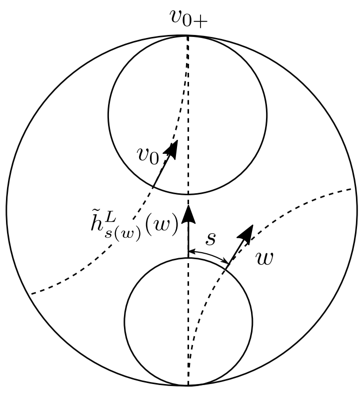

Figure 5.4: Definition of $s(w)$.

The problem is reduced now to showing that sets of the form $\tilde{h}^L_{[-S,S]}(L) \cap \tilde{\Sigma}$ have finite μ'-measure. Recall that our measure μ is the product of some measure $\mu_{\tilde{W}^{ws}(v_0)\cap\tilde{\Sigma}}$ on $\tilde{W}^{ws}(v_0) \cap \tilde{\Sigma}$ and the Lebesgue measures on the horocycles. Subsequently,

$$\mu(\tilde{h}^L_{[-S,S]}(L) \cap \tilde{\Sigma}) = 2S \cdot \mu_{\tilde{W}^{ws}(v)\cap\tilde{\Sigma}}(L \cap \tilde{\Sigma}).$$

Since $\tilde{h}_{[-S,S]}(L)$ is covered by a finite number of images of D by elements of Γ, and $D \cap \tilde{\Sigma}$ has finite μ-measure by hypothesis, then the left hand side of the previous equation is finite, and so is $\mu_{\tilde{W}^{ws}(v)\cap\tilde{\Sigma}}(L \cap \tilde{\Sigma})$.

The measure μ' of the set $\tilde{h}^L_{[-S,S]}(L) \cap \tilde{\Sigma}$ can also be decomposed as

$$\mu'(\tilde{h}^L_{[-S,S]}(L) \cap \tilde{\Sigma}) = \int_{L\cap\tilde{\Sigma}} \int_{s^M(v,-S)}^{s^M(v,S)} \mathrm{d}s \, \mathrm{d}\mu_{\tilde{W}^{ws}(v)\cap\tilde{\Sigma}}(u),$$

where s^M is the change of time between the Lebesgue and the Margulis flows. If the quantities $s^M(v,-S)$ and $s^M(v,S)$ are bounded on the set $L\cap\tilde{\Sigma}$, then the last integral is bounded and we obtain the desired result.

We have the equality

$$s^M(v, \pm S) = \pm\mu_{\tilde{H}^u(v)}((v, \tilde{h}^L_{\pm S}(v))).$$

Thanks to the continuity of the measure on the horocycles, the functions $v \mapsto \mu_{\tilde{H}^u(v)}((v, \tilde{h}^L_{\pm S}(v)))$ are continuous (and globally defined). So they are bounded on L because of the compactness. This completes the proof of the first implication.

Let $\mu \in \mathrm{Mes}(\tilde{h}^L_s|_{\tilde{\Sigma}})$ any Γ-invariant measure and assume that its image $\mu' = \Phi(\mu)$ in $\mathrm{Mes}(\tilde{h}^M_s)$ induces a finite measure on the quotient $\Sigma \subset T^1M$. We need to show that the μ-measure of $D \cap \tilde{\Sigma}$ is finite, where D is a compact fundamental domain. Similarly to the first situation we can reduce the problem to showing that, if $S > 0$, L is a compact subset of $\tilde{W}^{ws}(v)$ and $v \in \tilde{\Sigma}$ is a vector whose stable manifold has no rank 2 vectors, then the sets of the form $\tilde{h}^L_{[-S,S]}(L) \cap \tilde{\Sigma}$ have finite μ-measure. Since

$$\mu(\tilde{h}^L_{[-S,S]}(L) \cap \tilde{\Sigma}) = 2S \cdot \mu_{\tilde{W}^{ws}(v)\cap\tilde{\Sigma}}(L \cap \tilde{\Sigma}),$$

this is equivalent to showing that $L \cap \tilde{\Sigma}$ has finite $\mu_{\tilde{W}^{ws}(v)\cap\tilde{\Sigma}}$-measure.

We know that $\tilde{h}^L_{[-S,S]}(L) \cap \tilde{\Sigma}$ has finite μ'-measure, because it can be covered by finitely many images of D, and $D \cap \tilde{\Sigma}$ has finite measure. This measure is the integral of the function $s^M(v,S) - s^M(v,-S)$ over $L\cap\tilde{\Sigma}$ with respect to $\mu_{\tilde{W}^{ws}(v)\cap\tilde{\Sigma}}$. But the function $s^M(v,S) - s^M(v,-S)$ is strictly positive because $S > 0$, so the measure $\mu_{\tilde{W}^{ws}(v)\cap\tilde{\Sigma}}(L\cap\tilde{\Sigma})$ is finite, otherwise we would obtain an infinite integral. This proves that $\mu(\tilde{h}^L_{[-S,S]}(L))$ is finite, as we wanted. □

Corollary 5.2.6. *Let M be an oriented rank 1 compact connected Riemannian surface with nonpositive curvature and let Σ be a subset of T^1M whose lift $\tilde{\Sigma}$ to $T^1\tilde{M}$ satisfies the hypothesis of Proposition 5.2.4. Then the map $\mu \mapsto \Phi(\mu)/\Phi(\mu)(\Sigma)$ is a bijection between $\mathrm{Prob}(h^L_s|_\Sigma)$ and $\mathrm{Prob}(h^M_s)$.*

We finally apply this result to the subset Σ_0 of T^1M defined as the union of horocycles containing a rank 1 g_t-recurrent vector,

$$\Sigma_0 = \cup_{v\in Rec\cap R_1} H^u(v).$$

This set satisfies the hypothesis of Proposition 5.2.4, so the Margulis parametrization of the horocyclic flow can be defined. In Theorem 5.1.7, we proved that the Margulis flow on Σ_0 is uniquely ergodic. Thanks to the work in this section, we now know that the same holds for the Lebesgue parametrization.

Theorem 5.2.7. *Let M be an oriented rank 1 compact connected Riemannian surface with nonpositive curvature. In restriction to Σ_0, the flow given by the parametrization by the arc length of the horocycles is uniquely ergodic.*

5.3 Unique ergodicity on compact surfaces without flat strips

We now look back to Proposition 5.2.4. There seem to be two difficulties in defining the Margulis parametrization on a given horocycle. One is a rather technical difficulty, the fact that half-horocycles have infinite measure. We are not sure to what extent this could fail. Both in Section 5.1 and here, we have worked under hypothesis where it is true. In contrast, the fact that a horocycle H contains an interval of μ_H-zero measure is a clear obstruction to the definition of the parametrization on H. This in fact can only happen if the interval consists of rank 2 vectors. This phenomenon is produced by flat strips.

A *flat strip* on the universal cover is a totally geodesic submanifold isometric to the space $\mathbb{R} \times [0, r]$ for some $r > 0$. Given a horocycle H in $T^1\tilde{M}$, an interval $[v, w] \subset H$ consists only of rank 2 vectors if and only if the geodesics generated by the vectors in $[v, w]$ form a flat strip. In this case, we say that H cuts a flat strip.

In the following we prove that the Margulis parametrization is defined on the whole unit tangent bundle if M has no flat strips. With the help of this parametrization, we show that the horocyclic flow is uniquely ergodic using standard techniques.

Proposition 5.3.1. *Let M be an oriented nonpositively curved compact surface without flat strips. Then for every horocycle H in $T^1\tilde{M}$ the measure μ_H is of full support in H and the μ_H-measure of each half-horocycle is infinite.*

Proof. Every interval (v, w) on the horocycle H contains a rank 1 vector. Otherwise, all the vectors in $[v, w]$ would be of rank 2, so the curvature would vanish everywhere on the geodesics they generate, and v and w would bound a flat strip.

Since the rank 1 set is open, then (v, w) contains an interval of rank 1 vectors. Now we use the fact that μ_H is a positive function times the projection of σ_0 on the rank 1 vectors (Definition 3.4.1). Recall that σ_0 is supported on the limit set, which is the whole boundary $\partial\tilde{M}$ because M is compact, so the measure of (v, w) is strictly positive. This proves that μ_H is fully supported.

To prove the infiniteness of the measures μ_H on half-horocycles, we consider the map from T^1M to $\mathbb{R}$ which sends v to the $\mu_{H^u(v)}$-measure of the horocyclic ball of center v and radius 1. This map is continuous, by Proposition 5.1.1, and T^1M is compact, so it attains an absolute minimum $\alpha \in \mathbb{R}$. But the map is everywhere strictly positive, so the constant α is strictly positive too. Now, a half-horocycle $H_{R/L}$ in the tangent space of the universal cover $\tilde{M}$, contains infinitely many disjoint unstable balls of radius 1, each of them with measure at least $\alpha > 0$. We conclude that the μ_H-measures of the half-horocycles $H_{R/L}$ are infinite. □

This result ensures that the conditions of Proposition 5.2.4 are satisfied on the whole unit tangent bundle. The Margulis parametrization can be defined everywhere for the class of nonpositively curved compact surfaces without flat strips. It induces a horocyclic flow h_s^M on T^1M. We deduce the main result of this section, namely the unique ergodicity of the horocyclic flow, thanks to the good properties of this parametrization.

Theorem 5.3.2. *Let M be an orientable nonpositively curved compact surface without flat strips. Then there is a unique Borel probability measure on T^1M*

invariant by h_s^M. This measure is a constant multiple of the Bowen-Margulis measure.

Proof. We observe that the Bowen-Margulis measure μ_{BM} is invariant by both the geodesic flow g_t and the Margulis horocyclic flow h_s^M. The expanding property of the measures on the horocycles translates to the commuting property $g_t \circ h_s^M = h_{se^{\delta t}}^M \circ g_t$ between the flows. In this situation there is an argument of Coudène to show the unique ergodicity of h_s^M when μ_{BM} is absolutely continuous with respect to the weak stable foliation [Cou09]. The geodesic flow of a nonpositively curved surfaces without flat strips does not meet this last requirement because there is not a local product structure on non-hyperbolic regions. Indeed, weak stable manifolds are tangent to unstable horocycles on rank 2 vectors. We thus need to adapt Coudène's argument.

Fortunately, the absolute continuity of μ_{BM} with respect to the weak stable foliation only intervenes in the proof of the equicontinuity of averages along a horocycle pushed by the geodesic flow. The latter fact can be shown in our case using the disintegration of μ_{BM} on the boundary $\partial\tilde{M}$.

Lemma 5.3.3. *Let M be an orientable nonpositively curved compact surface without flat strips. Let $f : T^1M \to \mathbb{R}$ be a continuous function. For every $v \in T^1M$ and $R > 0$, write*

$$M_R(f)(v) := \frac{1}{R}\int_0^R f(h_s^M(v))ds.$$

Then the family of functions $\{M_1(f \circ g_t)\}_{t>0}$ is equicontinuous at every point $v \in T^1M$.

Proof. We will rather work on the universal cover. Let f be a real continuous Γ-invariant function on $T^1\tilde{M}$. Since Γ is cocompact, the absolute value of f is bounded by a real constant $C > 0$. From the definition of $\tilde{h}_s^M$, the average of f is

$$M_1(f \circ g_t)(v) = \int_{[v,\tilde{h}_1^M(v)]} f \circ g_t \,\mathrm{d}\mu_{\tilde{H}^u(v)},$$

which is also written as

$$M_1(f \circ g_t)(v) = \int_{\partial\tilde{M}} \mathbf{1}_{[v_+,\tilde{h}_1^M(v)_+]\setminus\tilde{S}_+} \cdot f \circ g_t \circ P^{-1}_{\tilde{H}^u(v)} \cdot \phi_{\tilde{H}^u(v)} \,\mathrm{d}\sigma_0$$

by Lemma 4.2.3.

Given two vectors $v, w \in T^1\tilde{M}$, we observe

$$\begin{aligned}
&|M_1(f \circ g_t)(v) - M_1(f \circ g_t)(w)| \leq \\
&\quad\leq \int_{\partial\tilde{M}} |(\mathbf{1}_{[v_+,\tilde{h}_1^M(v)_+]\setminus\tilde{S}_+}\phi_{\tilde{H}^u(v)} - \mathbf{1}_{[w_+,\tilde{h}_1^M(w)_+]\setminus\tilde{S}_+}\phi_{\tilde{H}^u(w)}) \cdot f \circ g_t \circ P^{-1}_{\tilde{H}^u(w)}| \,\mathrm{d}\sigma_0 \\
&\quad+ \int_{\partial\tilde{M}} |\mathbf{1}_{[v_+,\tilde{h}_1^M(v)_+]\setminus\tilde{S}_+}\phi_{\tilde{H}^u(v)} \cdot (f \circ g_t \circ P^{-1}_{\tilde{H}^u(v)} - f \circ g_t \circ P^{-1}_{\tilde{H}^u(w)})| \,\mathrm{d}\sigma_0 \\
&\quad\leq C\int_{\partial\tilde{M}} |\mathbf{1}_{[v_+,\tilde{h}_1^M(v)_+]\setminus\tilde{S}_+}\phi_{\tilde{H}^u(v)} - \mathbf{1}_{[w_+,\tilde{h}_1^M(w)_+]\setminus\tilde{S}_+}\phi_{\tilde{H}^u(w)}| \,\mathrm{d}\sigma_0 \qquad (*) \\
&\quad+ \int_{[v,\tilde{h}_1^M(v)]} |f \circ g_t - f \circ g_t \circ P^{-1}_{\tilde{H}^u(w)} \circ P_{\tilde{H}^u(v)}| \mathrm{d}\mu_{\tilde{H}^u(v)}. \qquad (**)
\end{aligned}$$

The term $(*)$ is independent of t and tends to 0 when w tends to v. This is because the function $\mathbf{1}_{[w_+,\tilde{h}_1^M(w)_+]\backslash\tilde{S}_+}\phi_{\tilde{H}^u(w)}$ converges almost surely to $\mathbf{1}_{[v_+,\tilde{h}_1^M(v)_+]\backslash\tilde{S}_+}\phi_{\tilde{H}^u(v)}$. The function $P^{-1}_{\tilde{H}^u(w)} \circ P_{\tilde{H}^u(v)}$ in the term $(**)$ is defined at least on the set $P^{-1}_{\tilde{H}^u(v)}(P_{\tilde{H}^u(w)}(\tilde{H}^u(w)) \setminus \tilde{S}_+)$, which has full measure. The map

$$(w,u) \mapsto P^{-1}_{\tilde{H}^u(w)} \circ P_{\tilde{H}^u(v)}(u) = \bar{P}^{-1}(w_-, u_+, \beta_{w_-}(0, \pi(w)))$$

is continuous on its domain.

Given $\delta > 0$, we can take w in a neighborhood of v such that the distance between u and $P^{-1}_{\tilde{H}^u(w)} \circ P_{\tilde{H}^u(v)}(u)$ is less than δ for all $u \in [v, \tilde{h}_1^M(v)]$ where the function is defined. Since u and $P^{-1}_{\tilde{H}^u(w)} \circ P_{\tilde{H}^u(v)}(u)$ are on the same weak stable manifold, the distance between them is non-increasing when we apply g_t. The distance between $g_t(u)$ and $g_t \circ P^{-1}_{\tilde{H}^u(w)} \circ P_{\tilde{H}^u(v)}(u)$ is therefore less than δ for all positive t. In this way, the term $(**)$ is bounded by the modulus of continuity $\omega_f(\delta)$ of f, which goes to 0 if $\delta \to 0$ because f is uniformly continuous. This proves the equicontinuity of the functions $\{M_1(f \circ g_t)\}_{t>0}$ at v.

□

It is straightforward to see that nonwandering rank 1 vectors are contained in the support of μ_{BM} from the definition. But every vector is nonwandering and the rank 1 set is dense in T^1M, because M is compact. So μ_{BM} is fully supported. It is also known that the horocyclic foliation is transitive [Ebe73a, Theorem 5.2]. These are the remaining ingredients of Coudène's theorem.

The rest of the proof goes verbatim to the one by Coudène. We explain it for the sake of completeness. We use the equicontinuity to apply the Arzelà-Ascoli theorem to $\{M_1(f \circ g_t)\}_{t>0}$. Hence, $\{M_1(f \circ g_t)\}_{t>0}$ is relatively compact in the space of continuous functions on T^1M endowed with the uniform topology. Let $\bar{f}$ be an accumulation point of this family, so there is a sequence $t_k \to +\infty$ such that $M_1(f \circ g_{t_k})$ converges to $\bar{f}$.

The commuting relation between g_t and h_s^M yields the formula

$$M_1(f \circ g_t) = M_{e^{\delta t}}(f) \circ g_t. \tag{5.5}$$

Moreover, by the Von Neumann ergodic theorem, $M_R(f)$ converges in $L^2(\mu_{BM})$ to an h_s^M-invariant function Pf. Combining both facts with the g_t-invariance of μ_{BM}, we have

$$||\bar{f} - Pf \circ g_{t_k}||_2 \leq ||M_1(f \circ g_{t_k}) - \bar{f}||_2 + ||M_{e^{\delta t_k}}(f) - Pf||_2.$$

This inequality implies that $\bar{f}$ is a L^2 limit of h_s^M-invariant functions. So $\bar{f}$ is continuous and h_s^M-invariant, and in fact it is constant since h_s^M has a dense orbit. This constant is $\int f \mathrm{d}\mu_{BM}$.

Since $\{M_1(f \circ g_t)\}_{t>0}$ has a unique accumulation point, the quantity $M_1(f \circ g_t)$ converges uniformly to this accumulation point when t goes to infinity. Using again (5.5) and the fact that the accumulation point is constant, we deduce that $M_R(f)$ converges uniformly to the same constant when R goes to infinity. This implies the unique ergodicity of h_s^M.

□

Finally, as a corollary of Theorem 5.3.2, we can deduce the unique ergodicity of the horocyclic flow for other parametrizations, for example by the arc-length.

Corollary 5.3.4. *Let M be an orientable nonpositively curved compact surface without flat strips. The Lebesgue horocyclic flow h_s^L on T^1M is uniquely ergodic.*

Chapter 6

Unique ergodicity of the horocyclic flow on compact surfaces without conjugate points

In the last chapter of this manuscript we use a more powerful approach to investigate the question of unique ergodicity of the horocyclic flow. This approach is based on the work of Gelfert and Ruggiero [GR19, GR20], who construct an expansive model of the geodesic flow on certain compact surfaces without conjugate points. We will explain how this model is also good to derive ergodic properties of the horocyclic flow. As we have seen in the previous chapter, the main difficulty in our way to study ergodic properties of the horocycles are strips. These strips are generated by nontrivial intersections of stable and unstable horocycles. A naive idea, but which indeed will work very well, is to collapse them, thus obtaining a quotient space, and study the dynamics there. We define a continuous flow on this new space with a uniformly expanding parametrization, which will contain essentially all the information concerning the horocyclic flow. We will prove that this quotient flow is uniquely ergodic thanks to the uniformly expanding parametrization, and then lift the result to the original horocyclic flow. Our results are explained in a recent article [BC22].

We detail the construction of the expansive model in Section 6.1 and prove some additional properties of the family of measures on the horocycles in Section 6.2. Once we have defined the uniformly expanding parametrization of the horocyclic flow in the quotient, we will prove that it is uniquely ergodic in Section 6.3. Finally, in Section 6.4 we will prove the technical result allowing us to lift unique ergodicity to the horocyclic flow on the unit tangent bundle of our manifold.

6.1 Quotient by strips

Let M be a compact surface without conjugate points and genus equal or higher than 2. Such a surfaces admits a metric of negative curvature, and therefore it satisfies the visibility property. Our goal is to study the dynamics of the geodesic flow and, especially, the horocyclic flow on T^1M. Our approach is inspired by two recent works of K. Gelfert and R. Ruggiero [GR19, GR20].

Recall that in $T^1\tilde{M}$ a strip is generated by a nontrivial intersection of the stable and the unstable horocycle of a vector $\tilde{v} \in T^1\tilde{M}$,

$$\tilde{I}(v) = \tilde{H}^u(v) \cap \tilde{H}^s(v).$$

For $v \in T^1M$, we define $I(v)$ and the projection $\tilde{I}(\tilde{v})$ by $\pi : \tilde{M} \to M$ for any lift $\tilde{v}$ of v. The set $I(v)$ is compact and, by Proposition 2.5.2, it is the image of a continuous curve. Since horocycles on T^1M have no self-intersections [Ebe77, Theorem 4.5], the curves $I(v)$ have no self-intersections either.

The existence of strips is the most obvious difference between the structure of strict negatively curved surfaces and the structure of the surface M with the present conditions. A way to simplify the dynamics of the geodesic flow on the strips is to identify them into single orbits as follows.

Definition 6.1.1. Let $\sim$ be the equivalence relation on T^1M defined by

$$v \sim w \iff w \in I(v).$$

The quotient of T^1M by this equivalence relation is denoted by $X = T^1M/\sim$, and quotient map by $\chi : T^1M \to X$. We also define a flow $\phi_t : X \to X$ by putting for $\theta \in X$, $t \in \mathbb{R}$,

$$\phi_t(\theta) := \chi(g_t(v)), \text{ where } v \text{ is any vector in the class } \theta.$$

Observe that the flow ϕ_t is well-defined, because g_t preserves the intervals $I(v)$, and continuous. By definition, χ is a semi-conjugation between g_t and ϕ_t. Similarly, we define the quotient $\tilde{X}$ of $T^1\tilde{M}$ by the intervals $\tilde{I}(v)$, and the quotient map $\tilde{\chi} : T^1\tilde{M} \to \tilde{X}$. all these objects are related as expressed in the following diagram.

$$\begin{tikzcd}
T^1\tilde{M} \arrow[r, "\tilde{\chi}"] \arrow[d, "d\pi"] & \tilde{X} \arrow[d] \\
T^1M \arrow[r, "\chi"] & X
\end{tikzcd}$$

Gelfert and Ruggiero obtained strong results about the structure of the quotient space and the dynamics of the quotient flow under a regularity assumption on Green subbundles. These subbundles are the graphs of the stable and the unstable solutions of the Ricatti matrix equation introduced by L. W. Green in [Gre58] for manifolds without conjugate points with curvature bounded below. Later, P. Eberlein used these bundles to characterize Anosov geodesic flows [Ebe73b].

Recall that R_1 is the subset of T^1M formed by the vectors where the Green subspaces are linearly independent and $\mathcal{E}$ is the subset of T^1M formed by the vectors v with trivial interval $I(v)$.

Theorem 6.1.1. *[GR20] Let M be a compact connected surface without conjugate points of genus greater than one and with continuous stable and unstable Green bundles.*

1. *The families H^s and H^u are continuous foliations of T^1M by C^1 curves which are tangent to the stable and the unstable Green bundles, respectively.*
2. *The rank 1 set R_1 is invariant, open, and dense in T^1M, and it is contained in the expansive set $\mathcal{E}$.*

3. *The quotient space X is a compact topological 3-manifold and the quotient flow Ψ is expansive, topologically mixing, and has a local product structure.*

The existence of expansive stable and unstable leaves is one of the key ideas that makes the previous theorem work. We state below the result, which is a generalization of Lemma 4.2.2.

Proposition 6.1.2. *[GR20, Proposition 3.6 and Corollary 3.7] Let M be a compact connected surface without conjugate points of genus greater than one. For every $v \in R_1$ that is forward g_t-recurrent, for every $w \in H^s(v)$ there exists a sequence $t_n \to +\infty$ such that*

$$d(g_{t_n}(v), g_{t_n}(w)) \xrightarrow[n\to+\infty]{} 0 \text{ and } g_{t_n}(v) \to v.$$

Moreover, if the Green bundles are continuous, we have

$$H^s(v) \subset R_1 \subset \mathcal{E}.$$

The analogous statement holds true for H^u as $t \to -\infty$.

6.2 Properties of the measures on the horocycles

6.2.1 Patterson-Sullivan measure

Let M be a compact higher genus surface without conjugate points. We recall briefly that there exists a Patterson-Sullivan measure σ_0 on the boundary at infinity $\partial\tilde{M}$. In Section 3.3, we defined the Bowen-Margulis measure in terms of σ_0. According to [CKW21, Theorem 5.6], the lift μ_{BM} of the measure of maximal entropy for the geodesic flow g_t on T^1M gives full measure to $\tilde{\mathcal{E}} \subset T^1\tilde{M}$ and in restriction to this set it satisfies

$$\mathrm{d}\mu_{BM}(v) = e^{2\delta\langle v_-, v_+\rangle_0}\mathrm{d}t\, \mathrm{d}\sigma_0(v_-)\, \mathrm{d}\sigma_0(v_+), \tag{6.1}$$

where $\mathrm{d}t$ stands for the Lebesgue measure on the g_t orbit going from v_- to v_+.

We start by proving that endpoints of strips have zero Patterson-Sullivan measure. When a vector is both in R_1 and g_t-recurrent, then its horocycles are entirely contained in R_1 by Proposition 6.1.2. Let Rec_1 be the set of vectors in R_1 which are forward and backward g_t-recurrent and let $\tilde{Rec}_1$ be its lift to $T^1\tilde{M}$. A subscript $_+$ on a subset of $T^1\tilde{M}$ denotes its projection to the boundary and a superscript ^{c} denotes its complement.

Lemma 6.2.1. *We have $\tilde{Rec}_{1+} \cap (\tilde{R}_1^c)_+ = \emptyset$.*

Proof. Let $\eta \in \tilde{Rec}_{1+}$ and take $v \in \tilde{Rec}_1$ such that $v_+ = \eta$. By Proposition 6.1.2, we know that $\tilde{H}^s(v)$ is contained in the rank 1 set, so $\tilde{W}^{ws}(v)$ is also in the rank 1 set by invariance. Then, since the set of vectors pointing positively to η is exactly $\tilde{W}^{ws}(v)$, η is not the endpoint of a vector in $\tilde{R}_1^c$. □

Let σ_0 be a Patterson-Sullivan measure on $\partial\tilde{M}$.

Lemma 6.2.2. *$\tilde{Rec}_{1+}$ has full σ_0-measure.*

Proof. The Bowen-Margulis measure μ_{BM} is ergodic and fully supported [CKW21]. Since R_1 is open and g_t-invariant it has full μ_{BM}-measure. By the Poincaré recurrence theorem, μ_{BM}-a.e. vector is forward and backward recurrent, which yields that Rec_1 has full measure, and so does $\tilde{Rec}_1$.

Now consider the subset $A := \{v \in T^1\tilde{M} \mid v_- \notin \tilde{Rec}_{1+}\}$ of $T^1\tilde{M}$. The expression of the Bowen-Margulis measure on $\tilde{\mathcal{E}}$ (Equation 6.1) implies

$$\mu_{BM}(A) = \int_{(\tilde{Rec}_{1+})^c} \int_{\partial\tilde{M}} \infty e^{\delta(\xi|\eta)_0} \mathrm{d}\sigma_0(\eta)\mathrm{d}\sigma_0(\xi),$$

so A is negligible in $T^1\tilde{M}$ if and only if $\tilde{Rec}_{1+}$ has full measure in $\partial\tilde{M}$. Finally, we can observe that

$$A = \{v \in T^1\tilde{M} \mid v_- \notin \tilde{Rec}_{1+}\} \subset \tilde{Rec}_1^c,$$

which implies that A is actually negligible. $\square$

6.2.2 Product structure of the measure of maximal entropy

In Section 3.4, we defined a family of measures on the horocycles of $T^1\tilde{M}$. Since the endpoints of strips have zero σ_0-measure, we can simplify the expression of the measure μ_H on a unstable horocycle $H = \tilde{H}^u(v)$ by restricting it to the vectors with trivial strips. This measure is given by

$$\mathrm{d}\mu_H(w) = e^{\delta b_w(0)}\mathrm{d}\sigma_0(w_+).$$

Similarly, the measures on the stable horocycles satisfy

$$\mathrm{d}\mu_{\tilde{H}^s(v)}(w) = e^{\delta b_{-w}(0)}\mathrm{d}\sigma_0(w_-). \tag{6.2}$$

For each $v \in T^1\tilde{M}$, we define a measure ν_v on the weak stable leaf $\tilde{W}^{ws}(v)$ of v by

$$\nu_v(A) = \int_{\mathbb{R}} e^{\delta t} \int_{\tilde{H}^s(g_t v)} \mathbf{1}_A(w)\mathrm{d}\mu_{\tilde{H}^s(g_t v)}(w)\,\mathrm{d}t.$$

This is not the usual Margulis measure on $\tilde{W}^{ws}(v)$ which is uniformly expanded. In fact, $g_{t*}\nu_v = \nu_v$ and $\nu_{g_t v} = e^{-\delta t}\nu_v$. We can recover the Bowen-Margulis measure from the product of measures on the horocycles.

Proposition 6.2.3. *For every vector $u \in T^1\tilde{M}$ and every Borel subset $A \subset T^1\tilde{M} \setminus \tilde{W}^{ws}(u)$, we have*

$$\mu_{BM}(A) = \int_{\tilde{H}^u(u)} \int_{\tilde{W}^{ws}(v)} \mathbf{1}_A(w)\,\mathrm{d}\nu_v(w)\,\mathrm{d}\mu_{\tilde{H}^u(u)}(v).$$

Proof. For $v \in T^1\tilde{M}$, $\eta \in \partial\tilde{M} \setminus (\mathcal{E}^c)_+$ with $\eta \neq v_+$ and $t \in \mathbb{R}$, let $w^v_{t,\eta}$ the unique vector in $\tilde{H}^s(g_t v)$ pointing negatively to η. Also, for $\xi \in \partial\tilde{M} \setminus (\mathcal{E}^c)_+$ let v_ξ be the unique vector lying in $\tilde{H}^u(u)$ pointing to ξ. Finally, let $w^\xi_{t,\eta} := w^{v_\xi}_{t,\eta}$.

We prove the proposition by carrying the following computations. First, we write the double integral in terms of the measure on the boundary. Then we apply the equality

$$\beta_\xi(0, \pi(v_\xi)) = \beta_\xi(0, \pi(w^\xi_{t,\eta})) - t$$

to simplify. Finally, we integrate in t the indicator function of A at the point $w^{\xi}_{t,\eta}$, which is exactly the Lebesgue measure on the geodesic (η,ξ) of the set A. We can restrict all the integrals to vectors in $\mathcal{E}$ or endpoints in $\partial\tilde{M}\setminus(\mathcal{E}^c)_+$ because they are of full measure each time, thus we do not need to worry about vectors contained in strips.

$$
\begin{aligned}
&\int_{\tilde{H}^u(u)}\int_{\tilde{W}^{ws}(v)\cap\mathcal{E}} \mathbf{1}_A(w)\,\mathrm{d}\nu_v(w)\,\mathrm{d}\mu_{\tilde{H}^u(u)}(v)\\
&\quad=\int_{\tilde{H}^u(u)}\int_{\mathbb{R}}\int_{\tilde{H}^s(g_tv)\cap\mathcal{E}} \mathbf{1}_A(w)e^{\delta t}\mathrm{d}\mu_{\tilde{H}^s(g_tv)}(w)\,\mathrm{d}t\,\mathrm{d}\mu_{\tilde{H}^u(u)}(v)\\
&\quad=\int_{\tilde{H}^u(u)}\int_{\mathbb{R}}\int_{\partial\tilde{M}\setminus(\mathcal{E}^c)_+} \mathbf{1}_A(w^v_{t,\eta})e^{\delta t}e^{\delta\beta_\eta(0,\pi(w^v_{t,\eta}))}\,\mathrm{d}\sigma_0(\eta)\,\mathrm{d}t\,\mathrm{d}\mu_{\tilde{H}^u(u)}(v)\\
&\quad=\int_{\partial\tilde{M}\setminus(\mathcal{E}^c)_+}\int_{\mathbb{R}}\int_{\partial\tilde{M}\setminus(\mathcal{E}^c)_+} \mathbf{1}_A(w^\xi_{t,\eta})e^{\delta t}e^{\delta\beta_\eta(0,\pi(w^\xi_{t,\eta}))}e^{\delta\beta_\xi(0,\pi(v_\xi))}\,\mathrm{d}\sigma_0(\eta)\,\mathrm{d}t\,\mathrm{d}\sigma_0(\xi)\\
&\quad=\int_{\partial\tilde{M}\setminus(\mathcal{E}^c)_+}\int_{\mathbb{R}}\int_{\partial\tilde{M}\setminus(\mathcal{E}^c)_+} \mathbf{1}_A(w^\xi_{t,\eta})e^{\delta\langle\eta,\xi\rangle_0}\,\mathrm{d}\sigma_0(\eta)\,\mathrm{d}t\,\mathrm{d}\sigma_0(\xi)\\
&\quad=\int_{\partial\tilde{M}\setminus(\mathcal{E}^c)_+}\int_{\partial\tilde{M}\setminus(\mathcal{E}^c)_+} e^{\delta\langle\eta,\xi\rangle_0}Leb_{\eta,\xi}(A)\,\mathrm{d}\sigma_0(\eta)\,\mathrm{d}\sigma_0(\xi)=\mu_{BM}(A).
\end{aligned}
$$

□

6.2.3 Additional properties of the measures on the horocycles

It is not hard to check that the family of measures $\{\mu_{\tilde{H}^u(v)}\}_{v\in T^1\tilde{M}}$ is Γ-invariant and is exponentially expanded by the geodesic flow,

$$\mu_{g_tH}=e^{\delta t}g_{t*}\mu_H,$$

because $b_{g_tv}(0)=t+b_v(0)$.

We can also show that the measures $\mu_{\tilde{H}^s(v)}$ defined by (6.2) vary continuously with v.

Proposition 6.2.4. *The map*

$$
\begin{array}{rcl}
\{(v,w)\in T^1\tilde{M}\times T^1\tilde{M}\mid w\in\tilde{H}^u(v)\} & \longrightarrow & \mathbb{R}\\
(v,w) & \longmapsto & \mu_{\tilde{H}^u(v)}((v,w))
\end{array}
$$

is continuous.

Proof. We want to show the continuity at (v,w), $w\in\tilde{H}^u(v)$. The map $(v',\eta)\in T^1\tilde{M}\times\partial\tilde{M}\mapsto\phi_{\tilde{H}^u(v')}(\eta)$ is continuous, so it is bounded by a constant C if we restrict v' to a small enough neighborhood of v and η to a relatively compact neighborhood of $[v_+,w_+]$ in $\partial\tilde{M}\setminus\{v_-\}$. The difference in measure with and interval (v',w') close to (v,w) is

$$|\mu_{\tilde{H}^u(v)}((v,w))-\mu_{\tilde{H}^u(v')}((v',w'))|\leq$$

$$\leq\int_{[v_+,w_+]}|\phi_{\tilde{H}^u(v)}-\phi_{\tilde{H}^u(v')}|\mathrm{d}\sigma_0+C\sigma_0([v_+,w_+]\triangle[v'_+,w'_+]).$$

The second term clearly goes to 0 when (v', w') approaches (v, w). In the first one, $\phi_{\tilde{H}^u(v')}$ converges pointwise to $\phi_{\tilde{H}^u(v)}$ when $v' \to v$. Moreover, $\phi_{\tilde{H}^u(v')}|_{[v_+,w_+]}$ are dominated by the constant C. So the first integral tends to 0 by dominated convergence.

□

To prove one of the next properties we will need the following lemma. For each $v \in T^1\tilde{M}$, we define $l(v) \in [0, +\infty]$ to be the length of the interval $\pi(\tilde{I}(v)) = \pi(\tilde{H}^s(v)) \cap \pi(\tilde{H}^u(v))$.

Lemma 6.2.5. *The function l is bounded by a constant $R > 0$.*

Proof. The function l is upper semi-continuous and Γ-invariant, so the claim is reduced to showing that l cannot take the value $+\infty$.

The fact that $l(v)$ is infinity implies that we can find vectors w in $\tilde{H}^u(v) \cap \tilde{H}^s(v)$ at arbitraly large Sasaki distance from v. As we now explain, this contradicts the following consequence of Morse's theorem [Mor24]: there is a constant $Q > 0$ which bounds the Haussdorff distance between any two biasymptotic geodesics. It is enough to consider $w \in \tilde{H}^u(v) \cap \tilde{H}^s(v)$ such that $d(\pi(v), \pi(w)) > 2Q$. The geodesics γ_v and γ_w are biasymptotic, but $d(\pi(v), \gamma_w(t)) \geq d(\pi(v), \pi(w)) - |t| > Q$ for $|t| \leq Q$ and $d(\pi(v), \gamma_w(t)) \geq |b^v(\gamma_w(t))| = |b^w(\gamma_w(t))| > Q$ for $|t| > Q$.

□

Finally, we can show the following properties of the measures on the horocycles.

Proposition 6.2.6. *For every leaf $H = \tilde{H}^u(v)$, the measure μ_H satisfies the following:*

1. *it has no point masses,*

2. *it is finite on compact subsets,*

3. *for every $w \in H$, $v \notin \operatorname{supp} \mu_H$ if and only if $w \in \operatorname{int} \tilde{I}(v)$,*

4. *it gives infinite measure to half-horocycles.*

Proof. 1. It is true because σ_0 has no point masses. The latter is a consequence of the shadow lemma [CKW21, Proposition 5.4(b)], which says that any point has neighborhoods of arbitrarily small measure.

2. It is true because ϕ_H is a continuous function on $\partial\tilde{M} \setminus \{v_-\}$, so it is bounded on compact subsets.

3. If $w \in H$ is in the interior of $\tilde{I}(w)$ with respect to the topology on the leaf H, since $\mu_H(\operatorname{int} \tilde{I}(w)) = 0$, we see that w is not in the support of μ_H. Conversely, suppose that there is an open interval containing w with zero μ_H-measure. If $u \in U \setminus \tilde{I}(w)$, then $u \notin H^s(w)$ and actually w_+ and u_+ are distinct. A simple computation,

$$\mu_H(U) \geq \int_{[w_+,u_+]} \phi_H(\eta) \mathrm{d}\sigma_0(\eta) > 0,$$

yields a contradiction. So we have proved that $U \subset \tilde{I}(w)$ and the statement follows.

4. Consider the horocyclic flow h_t with the Lebesgue parametrization. Let $R > 0$ be the bound obtained in 6.2.5. Assume that

$$\mu_H(\{h_t(v)\}_{t\geq 0}) = \sum_{k=0}^{+\infty} \mu_H([h_{2Rk}(v), h_{2R(k+1)}(v))) < +\infty.$$

Then $\mu_H([h_{2Rk}(v), h_{2R(k+1)}(v))) \to 0$ when $k \to +\infty$. Let w be an accumulation point of $h_{2Rk}(v)$. By continuity of the measure we get that $\mu_{H^u(w)}((w, h_{2R}(w))) = 0$. But this means that $(w, h_{2R}(w)) \subset I(w)$, so $l(w) \geq 2R$, which contradicts the lemma. $\square$

6.2.4 Measures on the quotient X

We now turn our attention to the quotient X, which is where the next arguments will take place. Recall that X is defined as the quotient of T^1M by an equivalence relation whose classes $[v]$ are the intervals $I(v)$. We can also define the quotient $\tilde{X}$ of $T^1\tilde{M}$ by the equivalence relation which identifies elements in the same interval $\tilde{I}(v)$. The group Γ acts naturally on $\tilde{X}$ since $\gamma\tilde{I}(v) = \tilde{I}(\gamma v)$, and X can be thought as its quotient space.

We now bring horocycles and their measures to the quotients. We write

$$V^u([v]) = \chi(H^u(v)),\ \tilde{V}^u([v]) = \chi(\tilde{H}^u(v))$$

for the quotient horocycles, and then we push the measures,

$$\mu_{V^s([v])} = \chi_*\mu_{H^u(v)},\ \mu_{\tilde{V}^s([v])} = \chi_*\mu_{\tilde{H}^u(v)}.$$

The curves $V^u(\theta)$ form a continuous foliation of X because the charts used in [GR20, Lemma 4.4] to show the topological structure of X are in fact foliated charts of V^u. Next we transfer the properties of the previous sections about the measures on the horocycles to the quotient.

Proposition 6.2.7. *1. For all $\theta \in \tilde{X}$, $\mu_{\tilde{V}^u(\theta)}$ has no point masses.*

2. For all $\theta \in \tilde{X}$, $\mu_{\tilde{V}^u(\theta)}$ is finite on compact subsets.

3. For all $\theta \in \tilde{X}$, $\operatorname{supp}\mu_{\tilde{V}^u(\theta)} = \tilde{V}^u(\theta)$.

4. For all $\theta \in \tilde{X}$, $\mu_{\tilde{V}^u(\theta)}$ gives infinite measure to half-horocycles.

5. For all $\theta \in \tilde{X}$, for all $t \in \mathbb{R}$, for all $\gamma \in \Gamma$, we have

$$\mu_{\tilde{V}^u(\phi_t(\theta))} = e^{\delta t}(\phi_t)_*\mu_{\tilde{V}^u(\theta)} \quad \mu_{\tilde{V}^u(\gamma(\theta))} = \gamma_*\mu_{\tilde{V}^u(\theta)}$$

.

6. The map

$$\begin{array}{ccc} \{(\theta,\beta) \in \tilde{X}\times\tilde{X} \mid \beta \in \tilde{V}^u(\theta)\} & \longrightarrow & \mathbb{R} \\ (\theta,\beta) & \longmapsto & \mu_{\tilde{V}^u(\theta)}((\theta,\beta)) \end{array}$$

is continuous.

Proof. 1. We have $\mu_{\tilde{V}^u([v])}([w]) = \mu_{\tilde{H}^u(v)}(\mathcal{I}(w)) = 0$, because $\mathcal{I}(w)$ projects to a single point in the boundary.

2. Since χ is proper, the preimage of a compact set K in $\tilde{V}^u([v])$ is compact and $\mu_{\tilde{V}^u([v])}(K) = \mu_{\tilde{H}^u(v)}(\chi^{-1}(K))$ is finite by Proposition 6.2.6.

3. Let U be an open nonempty subset of $\tilde{V}^u([v])$. Then $\chi^{-1}(U)$ is an open neighborhood of $\mathcal{I}(w)$ in $\tilde{H}^u(v)$, where $[w]$ is a point in U. By Proposition 6.2.6, the two ends of the interval $\mathcal{I}(w) \subset \chi^{-1}(U)$ are in the support of $\mu_{\tilde{H}^u(v)}$, hence $\mu_{\tilde{H}^u(v)}(\chi^{-1}(U)) = \mu_{\tilde{V}^u([v])}(U) > 0$.

4. The positive half-horocycle of θ is given by $\chi(\{h_t(v)\}_{t\geq 0})$, where v is any vector in θ. Then this implies

$$\mu_{\tilde{V}^u(\theta)}(\chi(\{h_t(v)\}_{t\geq 0})) = \mu_{\tilde{H}^u(v)}(\{h_t(v)\}_{t\geq 0} \cup \mathcal{I}(v)) = \mu_{\tilde{H}^u(v)}(\{h_t(v)\}_{t\geq 0}) = +\infty.$$

Analogously, we show that the negative half-leaf $\chi(\{h_t(v)\}_{t\leq 0})$ has infinite measure.

5. This follows directly from the corresponding properties for $\{\mu_{\tilde{H}^u(v)}\}_{v\in T^1\tilde{M}}$.

6. For every $\theta \in \tilde{X}$ and every $\beta \in \tilde{V}^u(\theta)$, note that $\mu_{\tilde{V}^u(\theta)}((\theta,\beta)) = \mu_{\tilde{H}^u(v)}((v,w))$ for any $v \in \theta$ and any $w \in \beta$.

We proceed by contradiction: suppose that the map is not continuous at (θ,β). Then there exists two sequences $(\theta_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ converging to θ and β, respectively, with $\beta_k \in \tilde{V}^u(\theta_k)$ and such that

$$\forall k \quad |\mu_{\tilde{V}^u(\theta)}((\theta,\beta)) - \mu_{\tilde{V}^u(\theta_k)}((\theta_k,\beta_k))| > \varepsilon > 0. \tag{6.3}$$

For each k select any $v_k \in \theta_k$ and $w_k \in \beta_k$. Up to taking a subsequence, we can assume that v_k converges to a vector $v \in T^1\tilde{M}$ and w_k converges to $w \in T^1\tilde{M}$. The continuity of the quotient map implies that $v \in \theta$ and $w \in \beta$. But now, by the observation made at the beginning and the continuity of the family of measures $\{\mu_{\tilde{H}^u(v)}\}_{v\in T^1\tilde{M}}$ proved in Proposition 6.2.4, we obtain

$$\mu_{\tilde{V}^u(\theta_k)}((\theta_k,\beta_k)) = \mu_{\tilde{H}^u(v_k)}((v_k,w_k)) \to \mu_{\tilde{H}^u(v)}((v,w)) = \mu_{\tilde{V}^u(\theta)}((\theta,\beta)).$$

This is in clear contradiction with Equation 6.3.

□

6.3 Unique ergodicity of the horocyclic flow on X

Now we define a horocyclic flow on $\tilde{X}$. First, we orient the horocycles in $T^1\tilde{M}$ thanks to the orientation of the surface, and we also get an orientation of the images of the horospheres in $\tilde{X}$. For each $\theta \in \tilde{X}$, define $h_s(\theta)$ as the unique point in the positive (resp. negative) sense of $\tilde{V}^u(\theta)$ such that $\mu_{\tilde{V}^u(\theta)}((\theta, h_s(\beta))) = |s|$ if s is a positive (resp. negative) real number.

Proposition 6.3.1. *The family of maps $h_s : \tilde{X} \to \tilde{X}$ is a continuous Γ-invariant flow which satisfies $\phi_t \circ h_s = h_{se^{\delta t}} \circ \phi_t$ and whose orbits are the curves $\tilde{V}^u(\theta)$. Moreover, h_s preserves the measure $\mu_{\tilde{X}} = \chi_*\mu_{BM}$.*

Proof. Since both ends of $\tilde{V}^u(\theta)$ have infinite measure, the point $h_s(\theta)$ exists, and it is unique because $\mu_{\tilde{V}^u(\theta)}$ has full support. Clearly, h_s is a flow. Let us prove that it is continuous. Let $(s_k, \theta_k) \to (s, \theta) \in \mathbb{R} \times T^1\tilde{M}$. We want to prove that $h_{s_k}(\theta_k)$ converges to $h_s(\theta)$. We assume $s \geq 0$, the other case being done analogously. We know that the curves $\tilde{V}^u(\beta)$ depend continuously on β, so there exist a sequence β_k converging to $h_s(\theta)$ such that $\beta_k \in \tilde{V}^u(\theta_k)$. By Proposition 6.2.7, we know that $\mu_{\tilde{V}^u(\theta_k)}((\theta_k, \beta_k))$ converges to $\mu_{\tilde{V}^u(\theta))}((\theta, h_s(\theta))) = s$ when k tends to infinity. We deduce then that the measures of the intervals $(\beta_k, h_{s_k}(\theta_k))$ go to 0.

We claim that the distance between β_k and $h_{s_k}(\theta_k)$ tends to 0. Assume, contrary to our claim, that, for some $\varepsilon > 0$, there is a subsequence k_i such that the distance $d(\beta_{k_i}, h_{s_{k_i}}(\theta_{k_i}))$ is greater than ε. The intervals $(\beta_{k_i}, h_{s_{k_i}}(\theta_{k_i}))$ are accumulating to some interval in $\tilde{V}^u(\theta)$ of length at least ε. But since the measure of $(\beta_{k_i}, h_{s_{k_i}}(\theta_{k_i}))$ tends to 0, the limiting interval should have zero measure, which is a contradiction because $\mu_{\tilde{V}^u(\theta_k)}$ has full support in $\tilde{V}^u(\theta)$. Finally, since β_k converges to $h_s(\theta)$, the sequence $h_{s_k}(\theta_k)$ also converges to this point.

The Γ-invariance and the uniformly expanding property of h_s follow from the corresponding properties for $\mu_{\tilde{V}^u(\theta)}$. Finally, the conditional measures of $\mu_{\tilde{X}}$ along horocyclic orbits are exactly the $\mu_{\tilde{V}^u(\theta)}$, and since h_s preserves each of these, it preserves the measure $\mu_{\tilde{X}}$.

□

In this way, we obtain a uniformly expanding continuous flow on the quotient X, which will also be denoted by h_s.

Proposition 6.3.2. *The flow h_s on X is uniquely ergodic.*

Proof. We apply the following theorem due to Coudène:

Theorem 6.3.3. *[Cou09] Let X be a compact metric space, g_t and h_s two continuous flows on X which satisfy the relation : $g_t \circ h_s = h_{se^{\delta t}} \circ g_t$. Let μ a Borel probability measure invariant under both flows, which is absolutely continuous with respect to W^{ws}, and with full support. Finally assume that the flow h_s admits a dense orbit. Then h_s is uniquely ergodic.*

The flows ϕ_t and h_s, with the parametrization that we have just given, are continuous, satisfy the relation of the theorem above (Proposition 6.3.1) and preserve the measure $\mu_X = \chi_* \mu_{BM}$. The local weak unstable manifolds W^{wu} of the quotient flow ϕ_t are the projections of the weak central leaves $\tilde{W}^{ws}$, proven in [GR20]. Moreover, the flow h_s is transversal to these manifolds and the flow ϕ_t has a product structure [GR20, Proposition 4.11]. Proposition 6.2.3 implies that the measure μ_X is locally the product of measures on the image of weak stable leaves by the measure on a curve V^u. In terms of the theorem, this means that μ_X is absolutely continuous with respect to W^{ws}. It also has full support [CKW21, Theorem 1.1]. Finally, the existence of a dense orbit of h_s follows from the minimality of the foliation H^u [Ebe77, Theorem 4.5]. □

6.4 Lift to the unit tangent bundle T^1M

So far, we have proved the unique ergodicity of a horocyclic flow on the quotient space X. In this final section, we will deduce that the horocyclic flow on the unit tangent bundle of the surface M with any parametrization is also uniquely ergodic. For this, we need a technical result allowing to relate the two flows from the point of view of invariant measures. This is done in Subsection 6.4.1, and then in Subsection 6.4.2 we apply the result to conclude the proof.

6.4.1 A general result on flow invariant measures

Let ϕ_t be a continuous flow without fixed points on a compact metric space X. We study the flow locally with the help of flow boxes.

Definition 6.4.1. An open subset U of X is called a *flow box* if there exists a closed subset T of U, $\varepsilon > 0$ and a homeomorphism

$$\Phi : T \times (-\varepsilon, \varepsilon) \longrightarrow U$$

such that $\Phi(x, s) = \phi_s(x)$ for all $(x, s) \in T \times (-\varepsilon, \varepsilon)$. The subset T is called a *transversal* and we write $U = \phi_{(-\varepsilon,\varepsilon)}(T)$ to express the fact that U is a flow box with transversal T.

The existence of transversals and flow boxes is guaranteed at the neighborhood of each point. We observe that if $U = \phi_{(-\varepsilon,\varepsilon)}(T)$ is a flow box, then for each open subset S of T and $0 < \varepsilon' < \varepsilon$, $U' = \phi_{(-\varepsilon',\varepsilon')}(S)$ is also a flow box. Since we can find arbitrarily small flow boxes containing any given point, they form a base of the topology. At some point it will be easier to work with a finite number of flow boxes which form a cover of the space; this is possible thanks to the compactness assumption.

Flow boxes describe the flow locally, but in order to recover the global dynamics of the flow, we will need extra information provided by the holonomies.

Definition 6.4.2. A *holonomy* is a map $\Theta : T_1 \to T_2$ between two transversals which is a homeomorphism onto its image, and such that, for every $x \in T_1$, $\Theta(x)$ lies in the orbit of x by ϕ_t.

We remark that holonomies exist locally, that is, given two transversals T_1 and T_2, if $x \in T_1$ and $\phi_t(x) \in T_2$ for some $t \in \mathbb{R}$, then for every $y \in T_1$ close to x there exists t_y, such that $y \mapsto t_y$ is continuous, with $t_x = t$ and such that $\phi_{t_y}(y) \in T_2$. So $\Theta(y) = \phi_{t_y}(y)$ is a holonomy.

Our goal is to study the measures preserved by the flow ϕ_t in terms of measures on the transversals of the flow.

Definition 6.4.3. A finite Borel measure μ on X is said to be *invariant* by the flow ϕ_t if $\mu(\phi_t(A)) = \mu(A)$ for all Borel subset A of X. Let $\{\mu_T\}_T$ be a family of finite Borel measures indexed on all possible transversals T to the flow ϕ_t, with each μ_T supported on the transversal T. The family $\{\mu_T\}_T$ is *invariant under holonomy* if every holonomy map $\Theta : T_1 \to T_2$ between two transversals preserves the measures, i.e. $\Theta_* \mu_{T_1} = \mu_{T_2}|_{\Theta(T_1)}$.

Proposition 6.4.1. *Let ϕ_t be a continuous flow without fixed points on a compact metric space X. There is a correspondence between finite Borel measures on X invariant by ϕ_t and families $\{\mu_T\}_T$ of finite Borel measures on the transversals invariant under holonomy.*

We only recall the main idea, the details can be found in [BS40, § 2]. If μ is invariant by ϕ_t, the measure μ_T of a Borel subset A of T can be defined as

$$\mu_T(A) = \mu(\phi_{(-\varepsilon,\varepsilon)}(A))/2\varepsilon$$

where ε is chosen small enough such that $\phi_{(-\varepsilon,\varepsilon)}(T)$ is a flow box. Conversely, the measure μ is defined from μ_T on a flow box $U = \phi_{(-\varepsilon,\varepsilon)}(T)$ as $\Phi_*(\mu_T \otimes Leb)$ where Φ is the homeomorphism of Definition 6.4.1 and *Leb* is the Lebesgue measure on the interval $(-\varepsilon, \varepsilon)$. One can check that these definitions do not depend on the choices made and that they glue up together, that the measures thus obtained satisfy the invariance properties, and that one construction is the inverse of the other.

We want to represent an invariant measure μ by their measures on the transversals μ_T. It is clear that a lot of information is redundant. For example, it would be enough to take the transversals associated to a finite cover by flow boxes. Then the measure on the rest of the transversals is recovered as push forwards from measures on the finite collection of transversals. In fact, we need a bit less.

Definition 6.4.4. A finite set of transversals $T_1, \dots, T_n$ will be called *complete* if every point in X is in the orbit of an element of one of the T_i's.

Proposition 6.4.2. *An invariant measure μ is determined by the measures μ_{T_1}, $\dots$, μ_{T_n} on a complete set of transversals $T_1, \dots, T_n$.*

Proof. In view of Proposition 6.4.1, we only need to show that the measures $\mu_{T_1}, \dots, \mu_{T_n}$ determine a family $\{\mu_T\}_T$ of measures on the transversals invariant under holonomy. We consider any transversal T and a point $y \in T$. Since the family of transversals is complete $y = \phi_t(x)$ for some $x \in T_i$. Now consider the holonomy $\Theta_{T_i,S}$ from a small neighborhood of x in T_i to a neighborhood S of y in T. The only way to define a measure on S that will be preserved by holonomy is to push forward the measure μ_{T_i} by the holonomy $\Theta_{T_i,S}$. The definition does not depend on the choice of T_i and x, because if we have a different holonomy $\Theta_{T_j,S}$ from a neighborhood in T_j to S, then $\Theta_{T_i,S*}\mu_{U_i} = (\Theta_{T_i,T_j} \circ \Theta_{T_j,S})_*\mu_{U_i} = \Theta_{T_j,S*}(\Theta_{T_i,T_j*}\mu_{U_i}) = \Theta_{T_j,S*}\mu_{U_j}$ thanks to the fact that the measures on T_i are preserved by holonomy. Moreover if we take another neighborhood S' of a possibly different point $y' \in T$, then the definitions of the measure coincide on $S \cap S'$ by a similar argument.

Consequently, we have a well defined family of measures $\{\mu_T\}_T$ on the transversals. It remains to show that they are invariant under holonomy. This immediately follows from the fact that the measures $\{\mu_{T_i}\}_{1\le i\le n}$ already have this property. □

We would like to remark that if the flow ϕ_t is minimal (that is, every orbit is dense in X), then any transversal T of ϕ_t is complete. Indeed, the orbit of any point $x \in X$ passes through an open flow box U around T, so the orbit intersects T and x is in the orbit of a point of T.

Finally we get at the main point of this section. We have seen that an invariant measure is determined by a finite number of transversal measures. So if we have two flows on two possibly different spaces, each with their transversals, but the dynamics on these transversals are essentially equal, then both flows will have the same invariant measures.

Definition 6.4.5. Let X, Y be two compact metric spaces and let $\phi_t : X \to X$, $\psi_t : Y \to Y$ be two continuous flows. Let $F : T \subset X \to Y$ a map defined on a subset T of X. We say that the map F preserves the holonomy if for all $x, y \in T$, x and y lie in the same orbit of ϕ_t if and only if $F(x)$ and $F(y)$ lie in the same orbit of ψ_t.

Theorem 6.4.3. *Let X, Y be two compact metric spaces and let $\phi_t : X \to X$, $\psi_t : Y \to Y$ be two continuous flows without fixed points. Let $(T_i)_{1\leq i\leq n}$, $(T'_i)_{1\leq i\leq n}$ be complete sets of transversals for the flows ϕ_t, ψ_t, respectively. Assume that there is a map $F : \sqcup_{i=1}^n T_i \to \sqcup_{i=1}^n T'_i$ which preserves the holonomy and such that for all $i \in \{1, \dots, n\}$, $F|_{T_i} : T_i \to F(T_i) = T'_i$ is a homeomorphism. Then there is a correspondence between the finite Borel measures on X invariant by ϕ_t and those on Y invariant by ψ_t.*

In particular, ϕ_t is uniquely ergodic if and only if ψ_t is uniquely ergodic.

Proof. Since invariant measures can be thought as families of measures $\{\mu_{T_i}\}_i$ and $\{\mu_{T'_i}\}_i$ on the complete sets of transversals invariant by holonomy, we only have to prove the correspondence between the latter objects. This correspondence is induced by the map F. Given a family $\{\mu_{T_i}\}_i$ of measures invariant under holonomy, we take the pushforward by F to obtain a family of measures $\{F_*\mu_{T_i}\}_i$ on the transversals T'_i. Let us check that they are also invariant under holonomy. Consider a holonomy $\Theta : U \subset T'_i \to T'_j$. Since Θ preserves the holonomy, the map $F^{-1} \circ \Theta \circ F : F^{-1}(U) \subset T_i \to T_j$ is a holonomy again. By the invariance of the first family of measures we have

$$(F^{-1} \circ \Theta \circ F)_*\mu_{T_i} = \mu_{T_j}|_{(F^{-1}\circ\Theta)(U)}.$$

Taking F_* we obtain

$$\Theta_*(F_*\mu_{T_i}) = (F_*\mu_{T_j})|_{\Theta(U)},$$

which proves the invariance of the $F_*\mu_{T_i}$. This process is reversible because Θ is inversible, so we have a bijection. □

This result is a generalization of Theorem 5.2.1, which corresponds to the case that one flow is a change of time of the other. The result also applies when the two flows are conjugate. But neither of these situations is the one where we will apply the theorem. There is no obvious relation between the horocyclic flows on the unit tangent bundle T^1M and on the quotient space X, in fact we do not even have a homeomorphism between both spaces.

6.4.2 Transversals to the horocyclic flows

Our final goal is to show the unique ergodicity of the horocyclic flow on T^1M. For this, we choose a continuous parametrization of the horocyclic foliation H^u, for example the parametrization h_t^L by the arc length of horocycles. We want to apply the result of the previous section to the flows h_t^L on T^1M and h_t on X. The natural projection $\chi : T^1M \to X$ is not well behaved with respect to these flows, it collapses segments and does no preserve the parametrizations. We will find complete sets of transversals for both flows such that the restriction of χ on these transversals satisfies the properties of Theorem 6.4.3.

Proposition 6.4.4. *There exists a transversal T of the flow h_t^L on T^1M such that $\chi(T)$ is a transversal of the flow h_t on X and $\chi : T \to \chi(T)$ is a homeomorphism which preserves the holonomy.*

Proof. The subset of generalized rank 1 vectors

$$R_1 = \{v \in T^1M \,|\, G^u(v) \neq G^s(v)\}$$

of T^1M is nonempty and open because we are assuming the continuity of the Green bundles. Moreover the tangent space to the stable (resp. unstable) leaf $H^s(v)$ is the stable (resp. unstable) Green subspace $G^s(v)$ at each point $v \in T^1M$. As a consequence, the weak stable leaf $W^{ws}(v)$ is transverse to the unstable leaf $H^u(v)$ at points $v \in R_1$. Fix $v \in R_1$, and consider a relatively compact neighborhood T of v in $W^{ws}(v) \cap R_1$, so the unstable leaves are still transverse to $W^{ws}(v)$ at each point of T.

The set R_1 does not contain vectors with nontrivial strips, so $\chi : R_1 \to \chi(R_1)$ is a homeomorphism as well as $\chi : T \to \chi(T)$. Since T is a transversal of the horocyclic flow h_t^L and $\bar{T} \subset R_1$, we can consider a flow box of the form $U = h^L_{(-\delta,\delta)}(T)$ included in R_1. Now, $\chi(T)$ is relatively compact and with closure included in the open subset $\chi(U)$. There exists $\varepsilon > 0$ such that $h_{(-\varepsilon,\varepsilon)}(\chi(T))$ is included in $\chi(U)$. It is straightforward to see that $h_{(-\varepsilon,\varepsilon)}(\chi(T))$ is a flow box with transversal $\chi(T)$ using the continuity of h_t.

Finally, χ automatically preserves the holonomy because for $v, w \in T$, we have $w \in H^u(v)$ if and only if $\chi(w) \in V^u(\chi(v)) = \chi(H^u(v))$. □

We recall that the flow h_t^L is minimal [Ebe77, Theorem 4.5]. Since minimality is a property of the orbits of the flows, the unstable leaves, and the map χ is continuous, it passes to the images of the unstable leaves, which are exactly the orbits of h_t. In short, the flow h_t on X is also minimal. As we observed after Proposition 6.4.2, in the case of minimal flows any transversal is complete in the sense of Definition 6.4.4, so we can directly apply Theorem 6.4.3 to the transversals T and $\chi(T)$ of the previous proposition and we get the correspondence between h_t^L-invariant measures and h_t-invariant measures. Since the flow h_t is uniquely ergodic, we get the desired result.

Theorem 6.4.5. *The horocyclic flow h_t^L on the unit tangent bundle of a compact surface of genus equal or higher than 2 without conjugate points and with continuous Green bundles is uniquely ergodic.*

Nomenclature

$\bar{\mu}$	Γ-invariant measure on $\partial^2\tilde{M}$
$\bar{P}$	Map from $T^1\tilde{M}$ to $\partial^2\tilde{M} \times \mathbb{R}$
β_ξ	Busemann cocycle
δ	Critical exponent
$\partial^2\tilde{M}$	Pairs of distinct points in $\partial\tilde{M}$
Γ	Isometry subgroup
Λ	Limit set of Γ
$\mathcal{E}$	Expansive set
μ_{BM}	Bowen-Margulis measure
Ω	Nonwandering set of g_t
$\partial\tilde{M}$	Boundary at infinity of $\tilde{M}$
Σ_0	Vectors whose horocycle contains a rank 1 recurrent vector
σ_0	Patterson-Sullivan measure
b_v	Busemann function
$E(\tilde{M})$	Pairs of geodesic endpoints
$G^s(v)$	Stable Green subspace of v
$G^u(v)$	Unstable Green subspace of v
g_t	Geodesic flow
$H^s(v)$	Stable horosphere of v in T^1M
$H^s(v)$	Unstable horosphere of v in T^1M
h_s	Horocyclic flow
$I(v)$	Intersection of the stable and the unstable horosphere
P	Projection from $T^1\tilde{M}$ to the endpoints
R_1	Rank 1 set
Rec	Set of g_t-recurrent vectors
c_v	Geodesic generated by a vector v
T^1M	Unit tangent bundle

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